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Slightly compressible and immiscible two-phase flow in porous media ☆

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ABSTRACT

We describe the flow of two compressible phases in a porous medium. We consider the case of slightly compressible phases for which the density of each phase follows an exponential law with a small compressibility factor. A nonlinear parabolic system including quadratic velocity terms is derived to describe compressible and immiscible two-phase flow in porous media. In one-dimensional space, we establish the existence and uniqueness of a local strong solution for the regularized system. We show also that the saturation is physically admissible. We describe the asymptotic behavior of the solutions when the compressibility factor goes to zero.

1. Introduction

Secondary recovery in petroleum engineering, unsaturated zone hydrology and soil science is a practical problem involving two-phase flow in a porous medium. Mathematical and numerical analyses of two-phase flow have been of interest for many years and there is an extensive body of literature on this subject. We merely mention a few references here.

The existence of solutions for immiscible incompressible two-phase flow in porous media has been analyzed [1–7]. Miscible compressible flow in porous media has also been investigated [8–14]. The situation is quite different for immiscible compressible two-phase flow in porous media, for which only a few results have been obtained. In 2004, a pioneering study investigated slightly compressible two-phase flow in a particular case in which the compressibility factors were the same for the two phase densities [15]. Galusinski and Saad analyzed some models for which they assumed the mass densities were bounded and depended on the Chavent global pressure [16,17]. The last condition was overcome in recent studies by assuming that each phase density depends on its own pressure and a more general immiscible compressible two-phase flow model in porous media was considered for fields with a single rock type [18,19]. Amaziane et al. introduced a new concept for global pressure for modeling immiscible compressible two-phase flows [20]. They obtained a nonlinear parabolic equation for the global pressure equation coupled to a nonlinear diffusion–convection equation for saturation.

In the case of a single phase, Douglas and Roberts considered miscible displacement of one compressible fluid and introduced numerical methods to approximate the solution [21]. For the same model, Amirat and Ziani established the existence of a global weak solution in a one-dimensional porous medium [10] and Feng studied the existence and uniqueness of a solution for a coupled system describing the single-phase miscible displacement of one compressible fluid in a one-dimensional porous medium [14,22].

Here we investigate compressible and immiscible two-phase flow in a one-dimensional porous medium. We assume that the mass density of each phase is unbounded and satisfies an exponential law. We formulate the system as a global pressure equation and a saturation equation. A common assumption in previous studies of immiscible and compressible two-phase

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flow is that the densities are increasing and bounded [16–20]; this assumption plays a major role in the multidimensional mathematical analysis.

Our interest in this study is in a slightly compressible fluid, and hence we assume that the density of each phase satisfies an exponential law with a small compressibility factor. We propose a formulation for compressible flow that involves an appropriate equation for the global pressure and one for saturation. We then prove the local existence and uniqueness of strong solutions for immiscible flows in a one-dimensional porous medium. We also present a maximum principle for saturation and describe the dependence of solutions on the compressibility factor. Finally, we establish uniform energy estimates with respect to the compressibility factor and show that the limit for compressible flow is the basic incompressible flow when the compressibility factor goes to zero.

2. Mathematical model

The equations describing the immiscible displacement of two compressible fluids are given by the following mass conservation of each phase:

$$\phi(x)\partial_t(\rho_i s_i)(t,x) + \mathrm{div}(\rho_i \mathbf{V}_i)(t,x) = 0, \quad i = 1, 2, \tag{2.1}$$

where ϕ is the porosity of the medium and ρ_i is the density and s_i the saturation of the ith fluid. The velocity of each fluid V_i is given by Darcy's law:

$$\mathbf{V}_i(t,x) = -\mathbf{K}(x)\frac{k_i(s_i(t,x))}{\mu_i}\nabla p_i(t,x), \quad i = 1, 2, \tag{2.2}$$

where $\mathbf{K}$ is the permeability tensor of the porous medium, k_i is the relative permeability, μ_i is the viscosity (considered to be constant) and p_i is the pressure of the ith phase. The gravity terms are not included since they do not affect the analysis result.

By the definition of saturation we have

$$s_1 + s_2 = 1, \tag{2.3}$$

where s_1 is gas saturation and s_2 is oil saturation. Thus, we define capillary pressure as

$$p_{12}(s_1) = p_1 - p_2. \tag{2.4}$$

The function $s_1 \longrightarrow p_{12}(s_1)$ is non-decreasing $\left(\frac{dp_{12}}{ds}(s_1) \geq 0 \text{ for all } s \in [0, 1]\right)$.

From these equations it is evident that the unknown functions are the saturation of one phase and one pressure.

We denote by

$$M_i(s_1) = \frac{k_i(s_1)}{\mu_i} \quad \text{and} \quad M(s_1) = M_1(s_1) + M_2(s_1)$$

the mobility of the ith phase and the total mobility. We can express the total velocity (i.e. the sum of the two velocities) in terms of p_2 and p_{12} [3]. We have

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = -\mathbf{K}M(s_1)\left(\nabla p_2 + \frac{M_1(s_1)}{M(s_1)}\nabla p_{12}(s_1)\right).$$

Finally, we denote by ν_i the fractional flow of the ith phase:

$$\nu_i(s_1) = \frac{M_i(s_1)}{M(s_1)}, \quad i = 1, 2,$$

and thus

$$\nu_1(s_1) + \nu_2(s_1) = 1 \quad \text{for all } s \in [0, 1].$$

It is possible to define a function $\tilde{p}(s_1)$ such that $\frac{d\tilde{p}}{ds_1}(s_1) = \nu_1(s_1)\frac{dp_{12}}{ds_1}(s_1)$. Setting $p = p_2 + \tilde{p}$, namely, global pressure [3], the total velocity becomes

$$\mathbf{V} = -\mathbf{K}M(s_1)\nabla p. \tag{2.5}$$

Thus, each phase velocity can be written as

$$\mathbf{V}_i = \nu_i\mathbf{V} - \mathbf{K}M\nu_1\nu_2\frac{dp_{12}}{ds_1}\nabla s_i, \quad i = 1, 2. \tag{2.6}$$

We denote $\alpha(s_1) = M(s_1)\nu_1(s_1)\nu_2(s_1)\frac{dp_{12}}{ds_1}(s_1) \geq 0$, and thus (2.1) can be rewritten as

$$\phi\partial_t(\rho_i s_i) + \mathrm{div}(\rho_i\nu_i\mathbf{V} - \mathbf{K}\rho_i\alpha(s_1)\nabla s_i) = 0, \quad i = 1, 2. \tag{2.7}$$

In previous studies an exponential state law was used to describe the displacement of one compressible fluid [10,21]. Following Aziz and Settari [23, p. 13], we consider that it is possible to assume that the fluid compressibility also follows an exponential law. The density of a fluid depends on its pressure, but taking advantage of the fact that this function varies slowly with capillary pressure [3, Chapter 4] we assume that $\rho_i = \rho_i(p)$ satisfies

$$\frac{d\rho_i}{dp}(p) = z_i \rho_i(p), \quad z_i > 0. \tag{2.8}$$

In this case, (2.7) can be reduced to

$$\phi \partial_t s_i + \phi s_i z_i \partial_t p + \mathrm{div}(\nu_i \mathbf{V}) + \nu_i z_i \mathbf{V} \cdot \nabla p - \mathrm{div}(\mathbf{K}\alpha \nabla s_i) - \mathbf{K}\alpha z_i \nabla s_i \cdot \nabla p = 0. \tag{2.9}$$

To simplify the notation, we write s for s_1 and ν for ν_1. To obtain a non-degenerate pressure equation, we add the two equations of system (2.9) to obtain the pressure equation

$$\phi(z_2 + (z_1 - z_2)s)\partial_t p + \mathrm{div}\, \mathbf{V} + (z_2 + (z_1 - z_2)\nu(s))\mathbf{V} \cdot \nabla p - \mathbf{K}\alpha(s)(z_1 - z_2)\nabla s \cdot \nabla p = 0. \tag{2.10}$$

Then Eq. (2.9) for $i = 1$ is considered to be the saturation equation

$$\phi \partial_t s + \phi s z_1 \partial_t p + \nu\, \mathrm{div}\, \mathbf{V} + \mathbf{V} \cdot \nabla \nu + \nu z_1 \mathbf{V} \cdot \nabla p - \mathrm{div}(\mathbf{K}\alpha \nabla s) - \mathbf{K}\alpha z_1 \nabla s \cdot \nabla p = 0. \tag{2.11}$$

To eliminate the second-order pressure term from (2.11), we replace $\mathrm{div}\, \mathbf{V}$ taken from (2.10) and find

$$\phi \partial_t s + \phi b(s) \partial_t p - \mathrm{div}(\mathbf{K}\alpha(s)\nabla s) + k(s)\mathbf{K}\nabla p \cdot \nabla p + a(s)\mathbf{K}\nabla s \cdot \nabla p = 0, \tag{2.12}$$

where

$$\begin{aligned} b(s) &= z_1 s - (z_2 + (z_1 - z_2)s)\nu(s), \\ k(s) &= -M(s)(z_1 \nu(s) - \nu(s)(z_2 + (z_1 - z_2)\nu(s))), \\ a(s) &= \nu(s)\alpha(s)(z_1 - z_2) - z_1 \alpha(s) - M(s)\frac{d\nu}{ds}(s). \end{aligned}$$

The system is a direct generalization of the incompressible model and is consistent with the incompressible model in the sense that it is the limiting form of the compressible system as the compressibility factors z_1 and z_2 of the fluids tend to zero. We prove this result in Section 8.

3. One-dimensional model

We are interested in a one-dimensional model of a porous medium model. The porous medium is considered to be homogeneous, so to simplify our notation we take $\mathbf{K}(x) \equiv I$ and $\phi(x) \equiv 1$. The constant compressibility factors z_i are considered to be such that $z_1 > z_2$. The existence and uniqueness of the solutions can be obtained in this general case, but our motivation is to study the behavior of the solution when the compressibility factors tend to zero. System (3.1)–(3.2) is obtained for $z_2 = \gamma > 0$, $z_1 = 2\gamma$, and for simplicity this system is expressed with one parameter γ independent of the saturation.

Let $T > 0$ be fixed and let $\Omega = (0, 1)$. We set $Q_T = \Omega \times (0, T)$. We investigate the following nonlinear boundary value problem that is parabolic in Ω_T:

$$\gamma d(s)\partial_t p - \partial_x(M(s)\partial_x p) - \gamma \beta(s)|\partial_x p|^2 - \gamma \alpha(s)\partial_x s \partial_x p = 0, \tag{3.1}$$

$$\partial_t s + \gamma b(s)\partial_t p - \partial_x(\alpha(s)\partial_x s) + \gamma k(s)|\partial_x p|^2 + a_\gamma(s)\partial_x s \partial_x p = 0, \tag{3.2}$$

where

$$\begin{aligned} &d(s) = (1 + s), \qquad \beta(s) = M(s)(1 + \nu(s)) \\ &b(s) = 2s - (1 + s)\nu(s), \qquad k(s) = -M(s)\nu(s)(1 - \nu(s)) \\ &\alpha(s) = M(s)\nu(s)(1 - \nu(s))\frac{dp_{12}}{ds}(s), \qquad a_\gamma(s) = \gamma \alpha(s)(\nu(s) - 1) - M(s)\frac{d\nu}{ds}(s). \end{aligned}$$

System (3.1)–(3.2) is coupled with the boundary conditions

$$\begin{cases} p(t, 0) = 1, & p(t, 1) = 0, \\ s(t, 0) = 0, & \alpha(s)\partial_x s(t, 1) = 0, \end{cases} \tag{3.3}$$

which model the injection of a fluid at a given pressure and the free production of fluids at s given pressure. To this system we add the initial conditions

$$s(0, x) = s_0(x), \qquad p(0, x) = p_0(x). \tag{3.4}$$

Next we make some assumptions for the coefficients in the system. In petroleum engineering, the total mobility $M(s)$ is positive and the fractional flow $\nu(s)$ is increasing and satisfies $0 \le \nu(s) \le 1$ for every $0 \le s \le 1$. For example, the Corey model [23] with $r_1 \geq 1$ and $r_2 \geq 1$ and relative permeabilities $k_1(s) = s^{r_1},\ k_2(s) = (1-s)^{r_2}$ gives

$$M(s) = \mu_1^{-1} s^{r_1} + \mu_2^{-1}(1-s)^{r_2}, \qquad \nu(s) = \frac{s^{r_1}}{s^{r_1} + \mu_1\mu_2^{-1}(1-s)^{r_2}}.$$

For convenience, we consider extensions of the functions d, β, b, k and a_γ on $\mathbb{R}$ such that

$$\begin{cases} d(s) \geq d_0 > 0, \qquad \|d\|_{W^{1,\infty}(\mathbb{R})} \leq c_0, \quad s \in \mathbb{R} \\ M(s) \geq m_0 > 0, \qquad \|M\|_{W^{1,\infty}(\mathbb{R})} \leq c_0, \quad s \in \mathbb{R} \\ \|b\|_{L^\infty(\mathbb{R})} + \|k\|_{L^\infty(\mathbb{R})} + \|\beta\|_{L^\infty(\mathbb{R})} \leq c_0, \\ \|a_\gamma\|_{L^\infty(\mathbb{R})} \leq c_0(1+\gamma), \\ \|\alpha\|_{W^{1,\infty}(\mathbb{R})} \leq c_0. \end{cases} \tag{3.5}$$

The capillary function $\alpha(s)$, which appears in the diffusion term $\partial_x(\alpha(s)\partial_x s)$, vanishes in general for $s = 1$ or/and $s = 0$, and thus the saturation equation (3.2) is a parabolic degenerate equation. This loss of coercivity is classical in porous media for incompressible [3,4] and compressible models [16–19].

Here we are interested in the non-degenerate case and we introduce a regularized problem in which α is replaced by $\bar{\alpha} = \alpha + \alpha_0$, $\alpha_0 > 0$. Then the function $\bar{\alpha}$ satisfies

$$\bar{\alpha}(s) \geq \alpha_0 > 0, \qquad \|\bar{\alpha}\|_{W^{1,\infty}(\mathbb{R})} \leq c_0, \quad s \in \mathbb{R}. \tag{3.6}$$

In one space dimension, we can give a representation of the boundary pressure. For this we denote

$$\pi(t,x) = p(t,x) - (1-x), \tag{3.7}$$

and thus the non-degenerate form of system (3.1)–(3.4) is written as

$$\gamma d(s)\partial_t \pi - \partial_x(M(s)(\partial_x \pi - 1)) - \gamma\beta(s)|\partial_x\pi - 1|^2 - \gamma\alpha(s)\partial_x s(\partial_x \pi - 1) = 0, \tag{3.8}$$

$$\partial_t s + \gamma b(s)\partial_t \pi - \partial_x(\bar{\alpha}(s)\partial_x s) + \gamma k(s)|\partial_x \pi - 1|^2 + a_\gamma(s)\partial_x s(\partial_x\pi - 1) = 0, \tag{3.9}$$

$$\pi(t,0) = \pi(t,1) = 0, \qquad s(t,0) = \partial_x s(t,1) = 0, \tag{3.10}$$

$$\pi(0,x) = \pi_0(x), \qquad s(0,x) = s_0(x). \tag{3.11}$$

Assuming $\beta = \alpha = 0$ in (3.8) and $k = 0$ in (3.9), the system is similar to that proposed by Douglas and Roberts [21] for which Feng [22] established the existence and uniqueness of a strong local time solution and Amirat and Ziani [10] proved the existence of a weak global solution in one space dimension. If the two phases have the same compressibility factor, the fourth term on the left-hand side vanishes (take $z_1 = z_2$ in (2.10), for example). By neglecting the third term on the left-hand side in (3.8), which is a quadratic pressure-gradient term, Galusinski and Saad authors analyzed three-dimensional flows [15]. Here, we are concerned with the complete system (3.8)–(3.11).

Our aims are to show that system (3.8)–(3.11) admits a unique strong solution, to describe the dependence of the solution with respect to the compressibility factor γ, and to establish that the incompressible model is the limiting form of the compressible system as γ tends to zero.

4. Main results

We introduce the following functional spaces:

$$V_0(\Omega) = \{u \in H^1(\Omega),\ u(0) = 0\}$$
$$W = \{u \in L^\infty((0,T); H_0^1(\Omega)) \cap L^2((0,T); H^2(\Omega)),\ \partial_t u \in L^2((0,T); L^2(\Omega))\}$$
$$V = \{u \in L^\infty((0,T); V_0(\Omega)) \cap L^2((0,T); H^2(\Omega)),\ \partial_t u \in L^2((0,T); L^2(\Omega))\}.$$

We now define weak and strong solutions and state our results.

Definition 4.1. Let $(\pi_0, s_0) \in H_0^1(\Omega) \times V_0(\Omega)$. A weak solution of (3.8)–(3.11) is a pair of functions (π, s) in $W \times V$ that satisfies

$$\int_{Q_T} \left(\gamma d(s)\partial_t\pi\phi + M(s)(\partial_x\pi - 1)\partial_x\phi - \gamma\beta(s)|\partial_x\pi - 1|^2\phi\right) dxdt$$
$$- \int_{Q_T} (\gamma\alpha(s)\partial_x s(\partial_x\pi - 1)\phi)\, dxdt = 0, \quad \forall\phi \in L^2((0,T); H_0^1(\Omega)) \tag{4.1}$$

$$\int_{Q_T} \left(\partial_t s\psi + \gamma b(s)\partial_t\pi\psi + \bar{\alpha}(s)\partial_x s\partial_x\psi + \gamma k(s)|\partial_x\pi - 1|^2\psi + a_\gamma(s)\partial_x s(\partial_x\pi - 1)\psi\right) dxdt = 0$$
$$\forall\psi \in L^2((0,T); V_0(\Omega)) \tag{4.2}$$

and

$$\pi(0,x) = \pi_0(x), \qquad s(0,x) = s_0(x). \tag{4.3}$$

Definition 4.2. Let $(\pi_0, s_0) \in H_0^1(\Omega) \times V_0(\Omega)$. We say that (π, s) is a strong solution of (3.8)–(3.11) if (π, s) belongs to $W \times V$ and satisfies Eqs. (3.8)–(3.11) almost everywhere in Q_T.

Classically, strong solutions are weak solutions and if the couple (π, s) is a strong solution, then the couple (p, s) is also a strong solution of system (3.1)–(3.4).

Theorem 4.1. *For every $\gamma > 0$, there exists a number $T_\gamma^* \in]0, T]$ such that system (3.8)–(3.11) has a strong solution on $Q_{T_\gamma^*}$. Moreover, if $0 \le s_0(x) \le 1$ a.e. in $x \in [0, 1]$, then $0 \le s(t, x) \le 1$ a.e. in $x \in [0, 1]$ for all $t \in [0, T_\gamma^*]$.*

In the proof of this theorem, we explicitly describe the dependence of solutions on the compressibility factor γ.

Theorem 4.2. *System (3.1)–(3.4) has a unique strong solution.*

Theorem 4.3. *The solution of system (3.8)–(3.11) converges to the solution of the usual incompressible model, obtained for γ equal to zero, as γ goes to zero.*

The remainder of the paper is organized as follows. Section 5 provides *priori* estimates of strong solutions that are used to derive bounds on the solutions and their gradients. In Section 6 we prove the existence of the solutions and show that saturation is physically relevant. In Section 7 we prove the uniqueness theorem. Section 8 provides more details about the statement of Theorem 4.3 and we study the asymptotic behavior of the solution when the compressibility factor goes to zero. Finally, Section 9 proves a technical result.

5. *A priori* estimates

In this section, we obtain several *a priori* estimates satisfied by the strong solution that lead to compactness properties essential in the proof. In the next proposition, we focus on the dependence of solutions on the compressibility factor.

Proposition 5.1. *Let (π, s) be a strong solution. Then (π, s) satisfies the following inequalities:*

$$\begin{aligned}\frac{d}{dt}\int_\Omega d(s)\left|\sqrt{\gamma}\pi\right|^2 dx + m_0\|\partial_x\pi\|^2_{L^2(\Omega)} \\ \le \|\partial_t s\|^2_{L^2(\Omega)} + c_1(1+\gamma)^3\left(1 + \|\partial_x s\|^6_{L^2(\Omega)} + \|\partial_x\pi\|^6_{L^2(\Omega)} + \left\|\sqrt{\gamma}\pi\right\|^6_{L^2(\Omega)}\right).\end{aligned} \tag{5.1}$$

$$\begin{aligned}d_0\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)} + \frac{d}{dt}\int_\Omega M(s)|\partial_x\pi - 1|^2 dx \le \|\partial_t s\|^2_{L^2(\Omega)} + \varepsilon_1\|\partial_{xx}\pi\|^2_{L^2(\Omega)} \\ + c_2(1+\gamma)^2(1 + \|\partial_x\pi\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)}).\end{aligned} \tag{5.2}$$

$$m_0\|\partial_{xx}\pi\|^2_{L^2(\Omega)} \le c_3\gamma\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)} + c_4(1+\gamma)^4(1 + \|\partial_x\pi\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)}). \tag{5.3}$$

$$\frac{d}{dt}\|s\|^2_{L^2(\Omega)} + \alpha_0\|\partial_x s\|^2_{L^2(\Omega)} \le c_5\gamma\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)} + c_6(1+\gamma)^2(1 + \|\partial_x\pi\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)} + \|s\|^6_{L^2(\Omega)}), \tag{5.4}$$

$$\begin{aligned}\|\partial_t s\|^2_{L^2(\Omega)} + \frac{d}{dt}\int_\Omega \bar\alpha(s)|\partial_x s|^2 dx \le \varepsilon_2\|\partial_{xx}s\|^2_{L^2(\Omega)} + \varepsilon_1\|\partial_{xx}\pi\|^2_{L^2(\Omega)} \\ + c_7\gamma\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)} + c_8(1+\gamma)^4(1 + \|\partial_x s\|^6_{L^2(\Omega)} + \|\partial_x\pi\|^6_{L^2(\Omega)}).\end{aligned} \tag{5.5}$$

$$\begin{aligned}\alpha_0\|\partial_{xx}s\|^2_{L^2(\Omega)} \le \frac{6}{\alpha_0}\|\partial_t s\|^2_{L^2(\Omega)} + \varepsilon_1\|\partial_{xx}\pi\|^2_{L^2(\Omega)} + c_9\gamma\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)} \\ + c_{10}(1+\gamma)^4(1 + \|\partial_x\pi\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)}),\end{aligned} \tag{5.6}$$

where $\varepsilon_1, \varepsilon_2$ are arbitrary positive constants and c_i is a constant independent of p, s and γ.

The proof of this proposition is given in Section 9. In the proof of Proposition 5.1, the one-dimensional properties of H^1 play a major role in obtaining the above inequalities. Thus, the extension of these results to the multi-dimensional case is not straightforward.

Proposition 5.2. *Let (π, s) be a strong solution. Then (π, s) satisfies the inequality*

$$\begin{aligned}\frac{d}{dt}\int_\Omega d(s)\left|\sqrt{\gamma}\pi\right|^2 dx + (1 + c_1\gamma)\frac{d}{dt}\int_\Omega M(s)|\partial_x\pi - 1|^2 dx + \frac{d}{dt}\|s\|^2_{L^2(\Omega)} \\ + c_2\frac{d}{dt}\int_\Omega \bar\alpha(s)|\partial_x s|^2 dx + m_0\|\partial_x\pi\|^2_{L^2(\Omega)} + m_0\|\partial_{xx}\pi\|^2_{L^2(\Omega)}\end{aligned}$$

$$+ d_0 \left\| \partial_t \left(\sqrt{\gamma}\pi \right) \right\|^2_{L^2(\Omega)} + \alpha_0 \|\partial_x s\|^2_{L^2(\Omega)} + \|\partial_t s\|^2_{L^2(\Omega)} + \alpha_0 \|\partial_{xx} s\|^2_{L^2(\Omega)}$$

$$\le c_3(1+\gamma)^4 \left(\left\| \sqrt{\gamma}\pi \right\|^6_{L^2(\Omega)} + \|\partial_x \pi\|^6_{L^2(\Omega)} + \|s\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)} + 1 \right), \quad (5.7)$$

where c_1, c_2 and c_3 are constants independent of π, s and γ.

Proof of Proposition 5.2. The estimate (5.7) is obtained in two steps. First, we eliminate $\|\partial_t s\|_{L^2(\Omega)}$ and $\left\| \partial_t \left(\sqrt{\gamma}\pi \right) \right\|_{L^2(\Omega)}$ on the right-hand side of (5.1)–(5.6). For that we multiply (5.5) by $\xi_1 = 3 + \frac{6}{\alpha_0}$ and (5.2) by $\xi_2 = 1 + \left(\frac{c_3}{d_0} + \frac{c_5}{d_0} + \xi_1 \frac{c_7}{d_0} + \frac{c_6}{d_0} \right) \gamma$ and we add estimates (5.1)–(5.6) to obtain

$$\frac{d}{dt} \int_\Omega d(s) \left| \sqrt{\gamma}\pi \right|^2 dx + m_0 \|\partial_x \pi\|^2_{L^2(\Omega)} + d_0 \left\| \partial_t \left(\sqrt{\gamma}\pi \right) \right\|^2_{L^2(\Omega)}$$

$$+ \xi_2 \frac{d}{dt} \int_\Omega M(s)|\partial_x \pi - 1|^2 dx + m_0 \|\partial_{xx}\pi\|^2_{L^2(\Omega)} + \frac{d}{dt} \|s\|^2_{L^2(\Omega)}$$

$$+ \alpha_0 \|\partial_x s\|^2_{L^2(\Omega)} + \|\partial_t s\|^2_{L^2(\Omega)} + \xi_1 \frac{d}{dt} \int_\Omega \bar{\alpha}(s)|\partial_x s|^2 dx + \alpha_0 \|\partial_{xx} s\|^2_{L^2(\Omega)}$$

$$\le \xi_1 \varepsilon_2 \|\partial_{xx} s\|^2_{L^2(\Omega)} + \varepsilon_1 (1 + \xi_1 + \xi_2) \|\partial_{xx}\pi\|^2_{L^2(\Omega)}$$

$$+ c(1+\gamma)^4 \left(1 + \left\| \sqrt{\gamma}\pi \right\|^6_{L^2(\Omega)} + \|\partial_x \pi\|^6_{L^2(\Omega)} + \|s\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)} \right).$$

By choosing ε_2 such that $\xi_1 \varepsilon_2 = \frac{\alpha_0}{2}$ and ε_1 such that $\varepsilon_1(1 + \xi_1 + \xi_2) = \frac{m_0}{2}$, we obtain (5.7). □

6. Proof of Theorem 4.1

In this section we prove the existence of solutions of system (3.8)–(3.11) and establish a maximum principle-type result for saturation.

6.1. Existence

We describe an essential consequence of Proposition 5.2 that leads to the local existence of the strong solution.

Proposition 6.1. *Let the assumptions of Proposition 5.1 be satisfied. Then for every $\gamma > 0$ there exists $T^*_\gamma \in (0, T)$ such that*

$$\|s\|_{L^\infty((0,T^*_\gamma);H^1(\Omega)) \cap L^2((0,T^*_\gamma);H^2(\Omega))} \le M(\gamma). \quad (6.1)$$

$$\|\partial_t s\|_{L^2((0,T^*_\gamma);L^2(\Omega))} \le M(\gamma). \quad (6.2)$$

$$\left\| \sqrt{\gamma}\pi \right\|_{L^\infty((0,T^*_\gamma);H^1(\Omega))} \le M(\gamma). \quad (6.3)$$

$$\left\| \sqrt{\gamma}\partial_t \pi \right\|_{L^2((0,T^*_\gamma);L^2(\Omega))} \le M(\gamma). \quad (6.4)$$

$$\|\partial_x \pi\|_{L^\infty((0,T^*_\gamma);L^2(\Omega))} + \|\partial_{xx}\pi\|_{L^2((0,T^*_\gamma);L^2(\Omega))} \le M(\gamma), \quad (6.5)$$

where $M(\gamma) = c(1+\gamma)^5$.

Proof. First, by Proposition 5.2 we know that (π, s) satisfies (5.7). We denote

$$g_\gamma(t) = \int_\Omega d(s) \left| \sqrt{\gamma}\pi \right|^2 dx + (1 + c_1\gamma) \int_\Omega M(s)d(s)|\partial_x \pi - 1|^2 dx + \int_\Omega s^2 dx + c_2 \int_\Omega \bar{\alpha}(s)|\partial_x s|^2 dx.$$

Hence,

$$g_\gamma(0) \le c(1+\gamma)(\|\pi_0\|^2_{H^1(\Omega)} + \|s_0\|^2_{H^1(\Omega)} + 1),$$

where c denotes a generic constant.

We then have

$$\left\| \sqrt{\gamma}\pi \right\|^6_{L^2(\Omega)} + \|\partial_x \pi - 1\|^6_{L^2(\Omega)} + \|s\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)}$$

$$\le \left(\left\| \sqrt{\gamma}\pi \right\|^2_{L^2(\Omega)} + \|\partial_x \pi - 1\|^2_{L^2(\Omega)} + \|s\|^2_{L^2(\Omega)} + \|\partial_x s\|^2_{L^2(\Omega)} \right)^3$$

$$\le c(g_\gamma(t))^3.$$

We define

$$F_\gamma(g_\gamma(t)) = c(1+\gamma)^4(g_\gamma(t))^3$$

and

$$k_\gamma(t) = c(1+\gamma)^4.$$

Then from (5.7) we deduce

$$g_\gamma(t) \le F_\gamma(g_\gamma(t)) + k_\gamma.$$

Using Simon's Lemma [24, p. 1098], we have that for every γ there exists T_γ^* such that

$$g_\gamma(t) \le g_\gamma(0) + \varepsilon = c(1+\gamma) \quad \forall t \in (0, T_\gamma^*).$$

We also have

$$\max_{0\le t\le T_\gamma^*}\left(\left\|\sqrt{\gamma}\pi\right\|^2_{L^2(\Omega)} + (1+c_1\gamma)\|\partial_x\pi - 1\|^2_{L^2(\Omega)} + \|s\|^2_{L^2(\Omega)} + \|\partial_x s\|^2_{L^2(\Omega)}\right) \le c(1+\gamma).$$

Integrating (5.7) with respect to t between 0 and T_γ^*, we find

$$m_0\|\partial_{xx}\pi\|^2_{L^2(\Omega_{T_\gamma^*})} + d_0\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega_{T_\gamma^*})} + \|\partial_t s\|^2_{L^2(\Omega_{T_\gamma^*})} + \alpha_0\|\partial_{xx}s\|^2_{L^2(\Omega_{T_\gamma^*})} \le c(1+\gamma)^5,$$

which completes the proof of Proposition 6.1. □

Corollary 6.1. *Assume $\gamma \le 1$. There exists $T^\star \in (0, T)$ independent of γ such that*

$$\|s\|_{L^\infty((0,T^*);H^1(\Omega))\cap L^2((0,T^*);H^2(\Omega))} \le c. \tag{6.6}$$

$$\|\partial_t s\|_{L^2((0,T^*);L^2(\Omega))} \le c. \tag{6.7}$$

$$\left\|\sqrt{\gamma}\pi\right\|_{L^\infty((0,T^*);H^1(\Omega))} \le c. \tag{6.8}$$

$$\left\|\sqrt{\gamma}\partial_t\pi\right\|_{L^2((0,T^*);L^2(\Omega))} \le c. \tag{6.9}$$

$$\|\partial_x\pi\|_{L^\infty((0,T^*);L^2(\Omega))} + \|\partial_{xx}\pi\|_{L^2((0,T^*);L^2(\Omega))} \le c. \tag{6.10}$$

Proof. Note that in view of the proof of Simon's lemma [24, p. 1098] and Proposition 6.1, the dependence of T_γ^* on γ is due to the dependence of F_γ and k_γ on γ. If $\gamma \le 1$, then $g_\gamma(0) \le g_0$, $|F_\gamma(y)| \le c|y|^3$ and $|k_\gamma| \le c$. Thus, the constant $M(\gamma)$ in (6.1)–(6.5) becomes independent of γ. □

According to Proposition 6.1, we can prove the existence of the solutions by constructing Galerkin-type approximations of (3.8)–(3.9) for pressure and the saturation. Next, we establish that (π, s) is actually a strong solution in a standard fashion. This is because π and $s \in L^2((0, T_\gamma^*); H^2(\Omega))$, which completes the proof of the existence part of Theorem 4.1.

6.2. *Admissibility of the solutions*

In this section we prove the admissibility of the solutions, that is, that the saturation remains in [0, 1]. Eqs. (2.10) and (2.11) with the simplification $z_2 = \gamma$, $z_1 = 2\gamma$, $\mathbf{K}(x) \equiv I$ and $\phi(x) \equiv 1$ is equivalent to system (3.1)–(3.2); by including the boundary conditions and the initial conditions, we obtain

$$\begin{cases} \gamma(1+s)\partial_t p + \partial_x V + \gamma(1+\nu(s))V\partial_x p - \gamma\alpha(s)\partial_x s\partial_x p = 0, \\ \partial_t s + 2\gamma s\partial_t p + \partial_x(\nu V) + 2\gamma\nu V\partial_x p - \partial_x(\bar{\alpha}(s)\partial_x s) - 2\gamma\alpha(s)\partial_x s\partial_x p = 0, \\ V = -M(s)\partial_x p \\ p(t,0) = 1, \qquad p(t,1) = 0, \qquad s(t,0) = 0, \qquad \partial_x s(t,1) = 0. \end{cases} \tag{6.11}$$

Now we express the extensions $\tilde{s}$, $\tilde{\nu}$, $\tilde{\alpha}$, which are assumed to be Lipschitz continuous on [0, 1], of the functions s, ν and α on $\mathbb{R}$ as

$$\tilde{s}(s) = s \quad \text{if } 0 \le s \le 1, \qquad \tilde{s}(s) = 0 \quad \text{if } s \le 0 \quad \text{and} \quad \tilde{s}(s) = 1 \quad \text{if } s \ge 1,$$
$$\tilde{\nu}(s) = \nu(s) \quad \text{if } 0 \le s \le 1, \qquad \tilde{\nu}(s) = 0 \quad \text{if } s \le 0 \quad \text{and} \quad \tilde{\nu}(s) = 1 \quad \text{if } s \ge 1,$$

and

$$\tilde{\alpha}(s) = \alpha(s) \quad \text{if } 0 \le s \le 1, \quad \text{and} \quad \tilde{\alpha}(s) = 0 \quad \text{elsewhere}.$$

The extension of M is continuous and satisfies $\tilde{M} \ge m_0 > 0$.

Note that this is the non-degenerate case and the function $s \longrightarrow \bar{\alpha}(s)$ in the term $\partial_x(\bar{\alpha}(s)\partial_x s)$is considered to satisfy (3.6). Thus, system (6.11) is

$$\begin{cases} \gamma(1+\tilde{s}(s))\partial_t p + \partial_x V + \gamma(1+\tilde{\nu}(s))V\partial_x p - \gamma\tilde{\alpha}(s)\partial_x s\partial_x p = 0, \\ \partial_t s + 2\gamma\tilde{s}(s)\partial_t p + \partial_x(\tilde{\nu}(s)V) + 2\gamma\tilde{\nu}(s)V\partial_x p - \partial_x(\bar{\alpha}(s)\partial_x s) - 2\gamma\tilde{\alpha}(s)\partial_x s\partial_x p = 0, \\ V = -\tilde{M}(s)\partial_x p \end{cases} \tag{6.12}$$

with

$$p(t,0) = 1, \qquad p(t,1) = 0, \qquad s(t,0) = 0, \qquad \partial_x s(t,1) = 0.$$

Multiplying the equation for saturation by $-s^- = \frac{s-|s|}{2}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|s^-\|^2_{L^2(\Omega)} - 2\gamma\int_\Omega \tilde{s}(s)\partial_t p s^- dx + \int_\Omega \tilde{\nu}(s)V\partial_x s^- dx - 2\gamma\int_\Omega \tilde{\nu}(s)V\partial_x p s^- dx$$
$$+ \int_\Omega \bar{\alpha}(s)|\partial_x(s^-)|^2 dx + 2\gamma\int_\Omega \tilde{\alpha}(s)\partial_x s\partial_x p s^- dx = 0.$$

Since $\tilde{s}(s) = \tilde{\nu}(s) = \tilde{\alpha}(s) = 0$ for $s \le 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|s^-\|^2_{L^2(\Omega)} + \int_\Omega \bar{\alpha}(s)|\partial_x(s^-)|^2 dx \le 0.$$

Integrating this inequality over $(0,t)$, we deduce $\|s^-(\cdot,t)\|_{L^2(\Omega)} \le \|s_0^-\|_{L^2(\Omega)}$ for all $t \in (0,T^*_\gamma)$ since $s_0 \ge 0$ in Ω, so then $s(\cdot,t) \ge 0$ in Ω for all $t \in (0,T^*_\gamma)$.

Multiplying the second equation of (6.12) by $(s-1)^+$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|(s-1)^+\|^2_{L^2(\Omega)} + 2\gamma\int_\Omega \tilde{s}(s)\partial_t p(s-1)^+ dx - \int_\Omega \tilde{\nu}(s)V\partial_x(s-1)^+ dx$$
$$- (\tilde{\nu}(s)V(s-1)^+)(t,1) + 2\gamma\int_\Omega \tilde{\nu}(s)V\partial_x p(s-1)^+ dx$$
$$+ \int_\Omega \bar{\alpha}(s)|\partial_x((s-1)^+)|^2 dx - 2\gamma\int_\Omega \tilde{\alpha}(s)\partial_x s\partial_x p(s-1)^+ dx = 0.$$

Using the fact that $\tilde{s}(s) = \tilde{\nu}(s) = 1$ for $s \ge 1$ and $\tilde{\alpha}(s) = 0$ for $s \ge 1$ the above equation is equivalent to

$$\frac{1}{2}\frac{d}{dt}\|(s-1)^+\|^2_{L^2(\Omega)} + 2\gamma\int_\Omega \partial_t p(s-1)^+ dx - \int_\Omega V\partial_x(s-1)^+ dx$$
$$- (V(s-1)^+)(t,1) + 2\gamma\int_\Omega V\partial_x p(s-1)^+ dx + \int_\Omega \bar{\alpha}(s)|\partial_x((s-1)^+)|^2 dx = 0. \tag{6.13}$$

Multiplying the pressure equation of system (6.12) by $(s-1)^+$, we have

$$\gamma\int_\Omega (1+\tilde{s}(s))\partial_t p(s-1)^+ dx + \int_\Omega \partial_x V(s-1)^+ dx + \gamma\int_\Omega (1+\tilde{\nu}(s))V\partial_x p(s-1)^+ dx$$
$$- \gamma\int_\Omega \tilde{\alpha}(s)\partial_x s\partial_x p(s-1)^+ dx = 0.$$

Using $\tilde{s}(s) = \tilde{\nu}(s) = 1$ for $s \ge 1$ and $\tilde{\alpha}(s) = 0$ for $s \ge 1$, we obtain

$$2\gamma\int_\Omega \partial_t p(s-1)^+ dx - \int_\Omega V\partial_x(s-1)^+ dx - (V(s-1)^+)(t,1) + 2\gamma\int_\Omega V\partial_x p(s-1)^+ dx = 0,$$

which leads, from (6.13), to

$$\frac{1}{2}\frac{d}{dt}\|(s-1)^+\|^2_{L^2(\Omega)} + \int_\Omega \bar{\alpha}(s)|\partial_x((s-1)^+)|^2 dx = 0.$$

Integrating this last inequality over $(0,t)$, we have $\|(s-1)^+(\cdot,t)\|_{L^2(\Omega)} \le \|(s_0-1)^+\|_{L^2(\Omega)}$ for all $t \in (0,T^*_\gamma)$, since $s_0 \le 1$ in Ω, so then $s(\cdot,t) \le 1$ in Ω for all $t \in (0,T^*_\gamma)$.

7. Uniqueness

In this section we prove Theorem 4.2. Let (p_1, s_1) and (p_2, s_2) be two strong solutions of system (3.1)–(3.3) corresponding to the same initial conditions (3.4). Then $q = p_1 - p_2$ and $u = s_1 - s_2$ satisfy

$$\begin{aligned}
&\gamma(1+s_1)\partial_t q + \gamma\partial_t p_2 u - \partial_x(M(s_1)\partial_x q) - \partial_x(\partial_x p_2(M(s_1) - M(s_2))) \\
&\quad - \gamma\beta(s_1)(\partial_x p_1 + \partial_x p_2)\partial_x q - \gamma|\partial_x p_2|^2(\beta(s_1) - \beta(s_2)) \\
&\quad - \gamma\alpha(s_1)\partial_x s_1\partial_x q - \gamma\alpha(s_1)\partial_x p_2\partial_x u - \gamma\partial_x p_2\partial_x s_2(\alpha(s_1) - \alpha(s_2)) = 0, \qquad (7.1)\\
&\partial_t u + \gamma b(s_1)\partial_t q + \gamma\partial_t p_2(b(s_1) - b(s_2)) - \partial_x(\bar{\alpha}(s_1)\partial_x u + \partial_x s_2(\bar{\alpha}(s_1) - \bar{\alpha}(s_2))) \\
&\quad + \gamma k(s_1)(\partial_x p_1 + \partial_x p_2)\partial_x q + \gamma|\partial_x p_2|^2(k(s_1) - k(s_2)) + a_\gamma(s_1)\partial_x s_1\partial_x q \\
&\quad + a_\gamma(s_1)\partial_x p_2\partial_x u + \partial_x p_2\partial_x s_2(a_\gamma(s_1) - a_\gamma(s_2)) = 0, \qquad (7.2)\\
&q(t,0) = q(t,1) = 0, \qquad u(t,0) = \partial_x u(t,1) = 0, \qquad (7.3)
\end{aligned}$$

and

$$u(0,x) = q(0,x) = 0. \qquad (7.4)$$

Thus, $1 + s_1 > 0$, and by replacing $\partial_t q$ by its value given by Eq. (7.1) in (7.2), we obtain

$$\begin{aligned}
&\partial_t u - \partial_x(\bar{\alpha}(s_1)\partial_x u) - \partial_x(\partial_x s_2(\bar{\alpha}(s_1) - \bar{\alpha}(s_2))) \\
&\quad + \frac{1}{1+s_1}\partial_x(M(s_1)\partial_x q) + \frac{1}{1+s_1}\partial_x(\partial_x p_2(M(s_1) - M(s_2))) + f_1(s_1)(\partial_x p_1 + \partial_x p_2)\partial_x q \\
&\quad + f_2(s_1)\partial_x s_1\partial_x q + f_3(s_1)\partial_x p_2\partial_x u + |\partial_x p_2|^2(h_1(s_1) - h_1(s_2)) + \partial_x p_2\partial_x s_2(h_2(s_1) - h_2(s_2)) \\
&\quad = \partial_t p_2(h_3(s_1) - h_3(s_2)), \qquad (7.5)
\end{aligned}$$

where the function $f_i\ (i = 1, 2, 3)$ is bounded in $L^\infty((0, T^*_\gamma); L^\infty(\Omega))$ and h_i, $i = 1, 2, 3$ is a Lipschitz function. We denote by $g(.)$ a function in $L^1(0, T^\star_\gamma)$ and by ε an arbitrary positive real number.

Multiplying Eq. (7.1) by q and integrating it over Ω, we have

$$\frac{\gamma}{2}\frac{d}{dt}\int_\Omega (1+s_1)q^2 dx + \int_\Omega M(s_1)|\partial_x q|^2 dx = J_1 + J_2 + J_3 + J_4 + J_5 + J_6, \qquad (7.6)$$

where

$$\begin{aligned}
J_1 &= \frac{\gamma}{2}\int_\Omega \partial_t s_1 q^2 dx,\\
J_2 &= -\gamma\int_\Omega \partial_t p_2 uq dx,\\
J_3 &= -\int_\Omega \partial_x p_2(M(s_1) - M(s_2))\partial_x q dx,\\
J_4 &= \gamma\int_\Omega (\beta(s_1)(\partial_x p_1 + \partial_x p_2) + \alpha(s_1)\partial_x s_1)\,\partial_x q q dx,\\
J_5 &= \gamma\int_\Omega \left(|\partial_x p_2|^2(\beta(s_1) - \beta(s_2)) - \partial_x p_2\partial_x s_2(\alpha(s_1) - \alpha(s_2))\right) q dx,\\
J_6 &= \gamma\int_\Omega \alpha(s_1)\partial_x p_2\partial_x u q dx.
\end{aligned}$$

The second term on the left-hand side is estimated as

$$\int_\Omega M(s_1)|\partial_x q|^2 dx \ge m_0\|\partial_x q\|^2_{L^2(\Omega)}.$$

We estimate each term on the right-hand side. From the first term we have

$$|J_1| \le \gamma\|\partial_t s_1\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}\|q\|_{L^\infty(\Omega)} \le g(t)\|q\|^2_{L^2(\Omega)} + \frac{m_0}{8}\|\partial_x q\|^2_{L^2(\Omega)}.$$

In the same way, the estimate of the second term yields

$$|J_2| \le \gamma\|\partial_t p_2\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\|q\|_{L^\infty(\Omega)} \le g(t)(\|u\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)}) + \frac{m_0}{8}\|\partial_x q\|^2_{L^2(\Omega)}.$$

Using the fact that the function M is Lipschitz, the third term gives

$$|J_3| \le c\|\partial_x p_2\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|\partial_x q\|_{L^2(\Omega)} \le g(t)\|u\|^2_{L^2(\Omega)} + \frac{m_0}{8}\|\partial_x q\|^2_{L^2(\Omega)}.$$

Taking again into account that $\partial_x s_i$, $\partial_x p_i$ $(i = 1,\ 2)$ belongs to $L^2((0, T^*_\gamma); L^\infty(\Omega))$, then the fourth term leads to the following estimate:

$$\begin{aligned}|J_4| &\le c(\|\partial_x p_1\|_{L^\infty(\Omega)} + \|\partial_x p_2\|_{L^\infty(\Omega)} + \|\partial_x s_1\|_{L^\infty(\Omega)})\|\partial_x q\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}\\ &\le g(t)\|q\|^2_{L^2(\Omega)} + \frac{m_0}{8}\|\partial_x q\|^2_{L^2(\Omega)}.\end{aligned}$$

The functions α and β are Lipschitz, so we obtain

$$\begin{aligned}|J_5| &\le c(\|\partial_x p_2\|^2_{L^\infty(\Omega)} + \|\partial_x p_2\|_{L^\infty(\Omega)}\|\partial_x s_2\|_{L^\infty(\Omega)})\|u\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}\\ &\le g(t)(\|u\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)}).\end{aligned}$$

The last term on the right-hand side of (7.6) is bounded by

$$|J_6| \le c\|\partial_x p_2\|_{L^\infty(\Omega)}\|\partial_x u\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)} \le g(t)\|q\|^2_{L^2(\Omega)} + \varepsilon\|\partial_x u\|^2_{L^2(\Omega)}.$$

Finally, from (7.6) we obtain the first estimate of interest, namely

$$\gamma\frac{d}{dt}\int_\Omega (1+s_1)q^2dx + m_0\|\partial_x q\|^2_{L^2(\Omega)} \le g(t)(\|u\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)}) + \varepsilon\|\partial_x u\|^2_{L^2(\Omega)}. \tag{7.7}$$

Multiplying Eq. (7.5) by u and integrating it over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \int_\Omega \bar\alpha(s_1)|\partial_x u|^2dx = I_1 + I_2 + I_3 + I_4 + I_5, \tag{7.8}$$

where

$$\begin{aligned}
I_1 &= \int_\Omega \left(\partial_x s_2(\bar\alpha(s_1) - \bar\alpha(s_2)) + \frac{1}{1+s_1}\partial_x p_2(M(s_1) - M(s_2)) + f_3(s_1)\partial_x p_2 u\right)\partial_x u dx,\\
I_2 &= \int_\Omega \left(\frac{1}{(1+s_1)^2}M(s_1)\partial_x s_1 + f_1(s_1)(\partial_x p_1 + \partial_x p_2) + f_2(s_1)\partial_x s_1\right)\partial_x q\ u dx,\\
I_3 &= \int_\Omega \left(|\partial_x p_2|^2(h_1(s_1) - h_1(s_2)) + \partial_x p_2\partial_x s_2(h_2(s_1) - h_2(s_2)) - \frac{1}{(1+s_1)^2}\partial_x p_2\partial_x s_1(M(s_1) - M(s_2))\right)u\, dx,\\
I_4 &= \int_\Omega \frac{1}{1+s_1}M(s_1)\partial_x q\partial_x u\, dx\\
I_5 &= \int_\Omega \partial_t p_2(h_3(s_1) - h_3(s_2))u\, dx.
\end{aligned}$$

The second term on the left-hand side of (7.8) is estimated as

$$\int_\Omega \bar\alpha(s_1)|\partial_x u|^2dx \ge \alpha_0\|\partial_x u\|^2_{L^2(\Omega)}.$$

Taking into account that the function $\bar\alpha$ and M are Lipschitz and $\partial_x s_i$, $\partial_x p_i$ belong to $L^2((0, T^*); L^\infty(\Omega))$, we have

$$|I_1| \le c(\|\partial_x s_2\|_{L^\infty(\Omega)} + \|\partial_x p_2\|_{L^\infty(\Omega)})\|\partial_x u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le g(t)\|u\|^2_{L^2(\Omega)} + \frac{\alpha_0}{6}\|\partial_x u\|^2_{L^2(\Omega)}.$$

In the same way, we have

$$\begin{aligned}|I_2| &\le c(\|\partial_x s_1\|_{L^\infty(\Omega)} + \|\partial_x p_1\|_{L^\infty(\Omega)} + \|\partial_x p_2\|_{L^\infty(\Omega)})\|\partial_x q\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\\ &\le g(t)\|u\|^2_{L^2(\Omega)} + \|\partial_x q\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the fact that h_i is also Lipschitz, we deduce that

$$\begin{aligned}|I_3| &\le c(\|\partial_x p_2\|^2_{L^\infty(\Omega)} + \|\partial_x p_2\|_{L^\infty(\Omega)}\|\partial_x s_2\|_{L^\infty(\Omega)} + \|\partial_x p_2\|_{L^\infty(\Omega)}\|\partial_x s_1\|_{L^\infty(\Omega)})\|u\|^2_{L^2(\Omega)}\\ &\le g(t)\|u\|^2_{L^2(\Omega)}.\end{aligned}$$

The estimates of I_4 and I_5 are classical. We obtain

$$|I_4| \le c\|\partial_x q\|_{L^2(\Omega)}\|\partial_x u\|_{L^2(\Omega)} \le c\|\partial_x q\|^2_{L^2(\Omega)} + \frac{\alpha_0}{6}\|\partial_x u\|^2_{L^2(\Omega)}$$

and

$$|I_5| \le c\|\partial_t p_2\|_{L^2(\Omega)}\|u\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)} \le g(t)\|u\|^2_{L^2(\Omega)} + \frac{\alpha_0}{6}\|\partial_x u\|^2_{L^2(\Omega)}.$$

Finally, from (7.8) we deduce the second estimate

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \alpha_0\|\partial_x u\|^2_{L^2(\Omega)} \le g(t)\|u\|^2_{L^2(\Omega)} + c\|\partial_x q\|^2_{L^2(\Omega)}. \tag{7.9}$$

To eliminate the gradient terms on the right-hand side of the estimates (7.7) and (7.9), we multiply (7.7) by $\left(1+\frac{c}{m_0}\right)$ and add it to (7.9). By choosing ε in (7.7) such that $\varepsilon\left(1+\frac{c}{m_0}\right) = \frac{\alpha_0}{2}$, we deduce

$$\begin{aligned}\frac{d}{dt}\int_\Omega (1+s_1)q^2dx + \frac{d}{dt}\|u\|^2_{L^2(\Omega)} &\le g(t)(\|u\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)})\\ &\le g(t)\left(\|u\|^2_{L^2(\Omega)} + \int_\Omega (1+s_1)q^2dx\right).\end{aligned}$$

Then the Gronwall lemma allows us to confirm that $u = q = 0$ in Ω for all $t \in (0, T^*_\gamma)$.

8. Compressible flow in the limit

In this section we study the behavior of the solution (π_γ, s_γ) for the problem (3.8)–(3.10) as γ goes to zero. For convenience, we recall from (4.1)–(4.2) the essential properties of the sequence $(\pi_\gamma, s_\gamma)_\gamma$:

$$\begin{aligned}&\int_{Q_{T^\star}} (\gamma d(s_\gamma)\partial_t\pi_\gamma\phi + M(s_\gamma)(\partial_x\pi_\gamma - 1)\partial_x\phi - \gamma\beta(s_\gamma)|\partial_x\pi_\gamma - 1|^2\phi\\ &\quad - \gamma\alpha(s_\gamma)\partial_x s_\gamma(\partial_x\pi_\gamma - 1)\phi)dxdt = 0,\end{aligned} \tag{8.1}$$

$$\begin{aligned}&\int_{Q_{T^\star}} \Bigg(\partial_t s_\gamma\psi + \gamma b(s_\gamma)\partial_t\pi_\gamma\psi + \bar\alpha(s_\gamma)\partial_x s_\gamma\partial_x\psi + \gamma k(s_\gamma)|\partial_x\pi_\gamma - 1|^2\psi\\ &\quad + \gamma\alpha(s_\gamma)(\nu(s_\gamma) - 1)\partial_x s_\gamma(\partial_x\pi_\gamma - 1)\psi - M(s_\gamma)\frac{d\nu}{ds}(s_\gamma)\partial_x s_\gamma(\partial_x\pi_\gamma - 1)\psi\Bigg)dxdt = 0\end{aligned} \tag{8.2}$$

for all $\phi \in L^2((0,T^*); H^1_0(\Omega))$, $\psi \in L^2((0,T^*); V_0(\Omega))$.

According to Corollary 6.1, the sequence $(\pi_\gamma, s_\gamma)_\gamma$ satisfies

$$\|\partial_t s_\gamma\|_{L^2((0,T^*);L^2(\Omega))} + \|s_\gamma\|_{L^\infty((0,T^*);H^1(\Omega))\cap L^2((0,T^*);H^2(\Omega))} \le c, \tag{8.3}$$

$$\left\|\sqrt{\gamma}\pi_\gamma\right\|_{L^\infty((0,T^*);H^1(\Omega))} + \left\|\sqrt{\gamma}\partial_t\pi_\gamma\right\|_{L^2((0,T^*);L^2(\Omega))} \le c, \tag{8.4}$$

$$\|\partial_x\pi_\gamma\|_{L^\infty((0,T^*);L^2(\Omega))} + \|\partial_{xx}\pi_\gamma\|_{L^2((0,T^*);L^2(\Omega))} \le c. \tag{8.5}$$

Proposition 8.1. *The sequence $(\pi_\gamma, s_\gamma)_\gamma$ converges, as γ goes to zero, to the unique classical limit (π, s) solution of the incompressible flow*

$$\int_{Q_{T^\star}} M(s)(\partial_x\pi - 1)\partial_x\phi dxdt = 0, \tag{8.6}$$

$$\int_{Q_{T^\star}} \partial_t s\psi + \bar\alpha(s)\partial_x s\partial_x\psi - M(s)\frac{d\nu}{ds}(s)\partial_x s(\partial_x\pi_\gamma - 1)\psi dxdt = 0 \tag{8.7}$$

for all $\phi \in L^2((0,T^*); H^1_0(\Omega))$, $\psi \in L^2((0,T^*); V_0(\Omega))$.

Proof. We denote by A^γ_1 to A^γ_4 the four integrals of (8.1) and by B^γ_1 to B^γ_6 the six integrals in (8.2). First, we show that A^γ_1, A^γ_3 and A^γ_4 go to zero as γ goes to zero.

From (8.4) we have

$$|A_1^\gamma| \le \sqrt{\gamma}\|d\|_{L^\infty(Q_{T^*})} \left\|\partial_t\left(\sqrt{\gamma}\pi_\gamma\right)\right\|_{L^2(Q_{T^*})} \|\phi\|_{L^2(Q_{T^*})} \le c\sqrt{\gamma}.$$

The third term in (8.1) can be rewritten as

$$A_3^\gamma = \int_{Q_{T^\star}} \left(\gamma\beta(s_\gamma)|\partial_x\pi_\gamma|^2\phi - 2\gamma\beta(s_\gamma)\partial_x\pi_\gamma\phi + \gamma\beta(s_\gamma)\phi\right) dxdt,$$

and integrating by parts the first term on the right-hand side, we have

$$\begin{aligned} A_3^\gamma = -\gamma\int_{Q_{T^\star}} &(\beta'(s_\gamma)\partial_x s_\gamma\partial_x\pi_\gamma\pi_\gamma\phi + \beta(s_\gamma)\partial_{xx}\pi_\gamma\pi_\gamma\phi + \beta(s_\gamma)\partial_x\pi_\gamma\pi_\gamma\partial_x\phi \\ &+ 2\beta(s_\gamma)\partial_x\pi_\gamma\phi - \beta(s_\gamma)\phi)dxdt. \end{aligned}$$

Considering the first term and taking into account (8.3)–(8.5), we have the following inequalities:

$$\begin{aligned} \left|\int_{Q_{T^\star}} \gamma\beta'(s_\gamma)\partial_x s_\gamma\partial_x\pi_\gamma\pi_\gamma\phi dxdt\right| &\le c\sqrt{\gamma}\|\beta'\|_{L^\infty(Q_{T^*})}\|\partial_x s_\gamma\|_{L^\infty((0,T);L^2(\Omega))} \\ &\quad\times \left\|\partial_t\left(\sqrt{\gamma}\pi_\gamma\right)\right\|_{L^\infty(Q_{T^*})}\|\partial_x\pi_\gamma\|_{L^2(Q_{T^*})}\|\phi\|_{L^2(0,T,L^\infty(\Omega))}. \end{aligned}$$

Similarly, we bound the rest of the terms for A_3^γ and obtain

$$|A_3^\gamma| \le c\sqrt{\gamma}.$$

Using the fact that $\partial_x s_\gamma$ and $\partial_x\pi_\gamma$ are bounded in $L^\infty((0,T^*);L^2(\Omega))$, we deduce

$$|A_4^\gamma| \le c\gamma.$$

Now we consider the equation for saturation. As for the estimate of $|A_1^\gamma|$, we also have $|B_2^\gamma| \le c\sqrt{\gamma}$. In the same way, B_4^γ is estimated as A_3^γ by replacing β_γ by k_γ and ϕ by ψ. Using the straightforward upper bounds of A_4^γ, we have $|B_5^\gamma| \le c\sqrt{\gamma}$.

To complete the proof of this proposition, we take the limit of A_2^γ, B_1^γ, B_3^γ and B_6^γ as γ goes to zero.

The estimates (8.3)–(8.5) and a compactness embedding lemma due to Simon [25] imply that there exists a subsequence, still denoted by (π_γ, s_γ), and (π, s) such that

$$s_\gamma \longrightarrow s \quad \text{in } \mathcal{C}([0,T^*];L^2(\Omega)) \cap L^2((0,T^*);H^1(\Omega)) \text{ strongly}, \tag{8.8}$$

$$\partial_x\pi_\gamma \longrightarrow \partial_x\pi \quad \text{in } L^2((0,T^*);L^2(\Omega)) \text{ weakly}, \tag{8.9}$$

$$\partial_t s_\gamma \longrightarrow \partial_t s \quad \text{in } L^2((0,T^*);L^2(\Omega)) \text{ weakly}. \tag{8.10}$$

Using the convergence-dominated Lebesgue theorem, we conclude that $(M(s_\gamma)\partial_x\phi)_\gamma$ converges to $M(s)\partial_x\phi$ strongly in $L^2((0,T^*);L^2(\Omega))$ and consequently

$$A_2^\gamma \longrightarrow \int_{Q_{T^\star}} M(s)(\partial_x\pi - 1)\partial_x\phi dxdt.$$

In the same way, we have

$$B_3^\gamma \longrightarrow \int_{Q_{T^\star}} \bar{\alpha}(s)\partial_x s\partial_x\psi dxdt.$$

From (8.10), we have

$$B_1^\gamma \longrightarrow \int_{Q_{T^\star}} \partial_t s\psi dxdt.$$

Finally, from (8.8) we have $\partial_x s_\gamma \longrightarrow \partial_x s$ in $L^2((0,T^*);L^2(\Omega))$ strongly and using (8.9) we deduce that

$$B_6^\gamma \longrightarrow -\int_{Q_{T^\star}} M(s)\frac{d\nu}{ds}(s)\partial_x s(\partial_x\pi_\gamma - 1)\partial_x\psi dxdt,$$

which completes the proof of Proposition 8.1. □

9. Proof of Proposition 5.1

The proof of this proposition follows the same step as for [22, Proposition 3.1]. Hereafter, c denotes a constant that is independent of π, s and γ. We describe in detail how to obtain the estimates (5.1) and then we explain in brief how to obtain the estimates (5.2)–(5.6).

Since (π, s) is a strong solution, it satisfies (3.8)–(3.9) with boundary conditions (3.10). By multiplying (3.8) by π and integrating it over Ω, we obtain

$$\begin{aligned}\frac{\gamma}{2}\frac{d}{dt}\int_\Omega d(s)\pi^2 dx + \int_\Omega M(s)|\partial_x\pi|^2 dx &= \gamma\int_\Omega \alpha(s)\partial_x s\partial_x\pi\pi\, dx + \gamma\int_\Omega \beta(s)|\partial_x\pi|^2\pi\, dx \\ &\quad + \frac{\gamma}{2}\int_\Omega d'(s)\pi^2\partial_t s dx + \int_\Omega M'(s)\partial_x s\pi\, dx - \gamma\int_\Omega \beta(s)\pi\, dx \\ &\quad - 2\gamma\int_\Omega \beta(s)\partial_x\pi\pi\, dx - \gamma\int_\Omega \alpha(s)\partial_x s\pi\, dx. \end{aligned}\tag{9.1}$$

Now we identify a bound for each term on the right-hand side. To do that we use the following Gagliardo–Nirenberg inequalities: [26,27]:

$$\|u\|_{L^\infty(\Omega)} \le M\|\partial_x u\|^{\frac12}_{L^2(\Omega)}\|u\|^{\frac12}_{L^2(\Omega)}, \quad \text{for all } u \in H^1_0(\Omega), \tag{9.2}$$

$$\|\partial_x u\|_{L^\infty(\Omega)} \le M\left(\|\partial_{xx} u\|^{\frac12}_{L^2(\Omega)}\|\partial_x u\|^{\frac12}_{L^2(\Omega)} + \|\partial_x u\|_{L^2(\Omega)}\right), \quad \text{for all } u \in H^1_0(\Omega)\cap H^2(\Omega). \tag{9.3}$$

The second term on the left-hand side of (9.1) is estimated by

$$\int_\Omega M(s)|\partial_x\pi|^2 dx \ge m_0\|\partial_x\pi\|^2_{L^2(\Omega)}.$$

From the Cauchy–Schwarz inequality and inequality (9.2), the first term on the right-hand side of (9.1) is bounded as follows:

$$\begin{aligned}\left|\gamma\int_\Omega \alpha(s)\partial_x s\partial_x\pi\pi\, dx\right| &\le \gamma\|\alpha\|_{L^\infty(\Omega)}\|\partial_x s\|_{L^2(\Omega)}\|\partial_x\pi\|_{L^2(\Omega)}\|\pi\|_{L^\infty(\Omega)} \\ &\le \gamma c_0 M\|\partial_x s\|_{L^2(\Omega)}\|\partial_x\pi\|^{\frac32}_{L^2(\Omega)}\|\pi\|^{\frac12}_{L^2(\Omega)},\end{aligned}$$

where c_0 is the constant in (3.5). Using Young's inequality, we have

$$\left|\gamma\int_\Omega \alpha(s)\partial_x s\partial_x\pi\pi\, dx\right| \le \frac{m_0}{12}\|\partial_x\pi\|^2_{L^2(\Omega)} + c\gamma^3\left(\|\partial_x s\|^6_{L^2(\Omega)} + \left\|\sqrt{\gamma}\pi\right\|^6_{L^2(\Omega)}\right). \tag{9.4}$$

To estimate the second term on the right-hand side of (9.1), we use exactly the same type of estimates as for (9.4) by replacing $\partial_x s$ by $\partial_x\pi$. We then have

$$\left|\gamma\int_\Omega \beta(s)|\partial_x\pi|^2\pi\, dx\right| \le \frac{m_0}{12}\|\partial_x\pi\|^2_{L^2(\Omega)} + c\gamma^3\left(\|\partial_x \pi\|^6_{L^2(\Omega)} + \left\|\sqrt{\gamma}\pi\right\|^6_{L^2(\Omega)}\right).$$

For the third term on the right-hand side of (9.1), we have

$$\begin{aligned}\left|\frac{\gamma}{2}\int_\Omega d'(s)\pi^2\partial_t s dx\right| &\le \frac{\gamma}{2}\|d'\|_{L^\infty(\Omega)}\|\partial_t s\|_{L^2(\Omega)}\|\pi\|_{L^2(\Omega)}\|\pi\|_{L^\infty(\Omega)} \\ &\le \frac12\|\partial_t s\|^2_{L^2(\Omega)} + \frac14\gamma^2 c_0^2 M^2(\|\partial_x\pi\|_{L^2(\Omega)}\|\pi\|^3_{L^2(\Omega)}) \\ &\le \frac12\|\partial_t s\|^2_{L^2(\Omega)} + \frac{m_0}{12}\|\partial_x\pi\|^2_{L^2(\Omega)} + c\gamma\left\|\sqrt{\gamma}\pi\right\|^6_{L^2(\Omega)}.\end{aligned}$$

According to the Poincare inequality, the fourth term on the right-hand side of (9.1) satisfies

$$\left|\int_\Omega M'(s)\partial_x s\pi\, dx\right| \le \frac{m_0}{12}\|\partial_x\pi\|^2_{L^2(\Omega)} + c\|\partial_x s\|^2_{L^2(\Omega)}$$

and the fifth term leads to

$$\left|\gamma\int_\Omega \beta(s)\pi\, dx\right| \le \frac{m_0}{12}\|\partial_x\pi\|^2_{L^2(\Omega)} + c\gamma^2.$$

In the same way, the sixth term gives

$$\left|2\gamma\int_\Omega \beta(s)\partial_x\pi\,\pi\,dx\right| \le c\gamma\,\|\partial_x\pi\|^2_{L^2(\Omega)}.$$

Finally, the last term yields

$$\left|\gamma\int_\Omega \alpha(s)\partial_x s\pi\,dx\right| \le \gamma\,\|\alpha\|_{L^\infty(\Omega)}\|\partial_x s\|_{L^2(\Omega)}\|\pi\|_{L^2(\Omega)} \le \frac{m_0}{12}\|\partial_x\pi\|^2_{L^2(\Omega)} + c\gamma^2\|\partial_x s\|^2_{L^2(\Omega)}.$$

By substituting these estimates into (9.1), we obtain the first inequality, (5.1).

Now, testing (3.8) using $\partial_t\pi$ and the relation

$$-\int_\Omega \partial_x(M(s)(\partial_x\pi - 1))\partial_t\pi\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega M(s)|\partial_x\pi - 1|^2dx - \frac{1}{2}\int_\Omega M'(s)\partial_t s|\partial_x\pi - 1|^2dx,$$

we have

$$\gamma\int_\Omega d(s)|\partial_t\pi|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega M(s)|\partial_x\pi - 1|^2dx = \frac{1}{2}\int_\Omega M'(s)|\partial_x\pi - 1|^2\partial_t s\,dx$$
$$+\gamma\int_\Omega \beta(s)|\partial_x\pi - 1|^2\partial_t\pi\,dx + \gamma\int_\Omega \alpha(s)\partial_x s(\partial_x\pi - 1)\partial_t\pi\,dx. \tag{9.5}$$

The first term on the left-hand side of (9.5) is bounded by

$$\gamma\int_\Omega d(s)|\partial_t\pi|^2dx \ge d_0\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)}.$$

To estimate the first term on the right-hand side, we use inequality (9.3) and Young's inequality and obtain

$$\left|\frac{1}{2}\int_\Omega M'(s)|\partial_x\pi - 1|^2\partial_t s\,dx\right| \le \frac{1}{2}\|\partial_t s\|^2_{L^2(\Omega)} + \varepsilon_1\|\partial_{xx}\pi\|^2_{L^2(\Omega)} + c(1 + \|\partial_x\pi\|^6_{L^2(\Omega)}),$$

where ε_1 is some undetermined positive real number.

For the second term on the right-hand side, we obtain

$$\left|\gamma\int_\Omega \beta(s)|\partial_x\pi - 1|^2\partial_t\pi\,dx\right| \le \frac{d_0}{4}\|\partial_t\left(\sqrt{\gamma}\pi\right)\|^2_{L^2(\Omega)} + \varepsilon_1\|\partial_{xx}\pi\|^2_{L^2(\Omega)} + c(\gamma+\gamma^2)(\|\partial_x\pi\|^6_{L^2(\Omega)}) + c\gamma$$

and the last term is bounded as follows:

$$\left|\gamma\int_\Omega \alpha(s)\partial_x s(\partial_x\pi - 1)\partial_t\pi\,dx\right| \le \frac{d_0}{4}\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)} + \varepsilon_1\|\partial_{xx}\pi\|^2_{L^2(\Omega)}$$
$$+ c(\gamma^2+\gamma)(\|\partial_x\pi\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)}) + c\gamma.$$

Finally, taking into account the above estimates of each term on the right-hand side of (9.5), we establish (5.2).

To obtain an estimate of the second derivatives of pressure, rewriting (3.8) as

$$\gamma d(s)\partial_t\pi - M(s)\partial_{xx}\pi - M'(s)(\partial_x\pi - 1)\partial_x s - \gamma\beta(s)|\partial_x\pi - 1|^2 - \gamma\alpha(s)\partial_x s(\partial_x\pi - 1) = 0$$

and testing it using $(-\partial_{xx}\pi)$, we have

$$\int_\Omega M(s)|\partial_{xx}\pi|^2dx = \gamma\int_\Omega \partial_t\pi\,\partial_{xx}\pi\,dx - \int_\Omega (M'(s) + \gamma\alpha(s))(\partial_x\pi - 1)\partial_x s\partial_{xx}\pi\,dx$$
$$-\gamma\int_\Omega \beta(s)|\partial_x\pi - 1|^2\partial_{xx}\pi\,dx. \tag{9.6}$$

The first term on the left-hand side is bounded by

$$\int_\Omega M(s)|\partial_{xx}\pi|^2dx \ge m_0\|\partial_{xx}\pi\|^2_{L^2(\Omega)}.$$

From the Cauchy–Schwarz inequality, the first term on the right-hand side of (9.6) gives

$$\left|\gamma\int_\Omega \partial_t\pi\,\partial_{xx}\pi\,dx\right| \le \frac{m_0}{6}\|\partial_{xx}\pi\|^2_{L^2(\Omega)} + \frac{3}{2m_0}\gamma\left\|\partial_t\left(\sqrt{\gamma}\pi\right)\right\|^2_{L^2(\Omega)}.$$

Using again the inequality (9.3) and Young's inequality, we deduce a bound for the second term on the right-hand side of (9.6):

$$\left|\int_\Omega (M'(s) + \gamma\alpha(s))(\partial_x\pi - 1)\partial_x s\partial_{xx}\pi\,dx\right| \le \frac{m_0}{6}\|\partial_{xx}\pi\|^2_{L^2(\Omega)} + c(1+\gamma)^4(1 + \|\partial_x\pi\|^6_{L^2(\Omega)} + \|\partial_x s\|^6_{L^2(\Omega)}).$$

The last term on the right-hand side of (5.3) can be estimated in the same way as the above estimate by replacing $\partial_x s$ by $(\partial_x \pi - 1)$ and $\gamma\beta(s)$ by $(M'(s) + \gamma\alpha(s))$. We obtain

$$\left|\gamma \int_\Omega \beta(s)|\partial_x \pi - 1|^2 \partial_{xx}\pi \, dx\right| \le \frac{m_0}{6}\|\partial_{xx}\pi\|^2_{L^2(\Omega)} + c(1+\gamma)^4(1 + \|\partial_x \pi\|^6_{L^2(\Omega)}).$$

By combining the estimates from the right-hand side of (9.6), we obtain the inequality (5.3). The estimates (5.4)–(5.6) are obtained in the same way as (5.1)–(5.3).

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