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STUDY OF DEGENERATE PARABOLIC SYSTEM MODELING THE HYDROGEN DISPLACEMENT IN A NUCLEAR WASTE REPOSITORY

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Abstract. Our goal is the mathematical analysis of a two phase (liquid and gas) two components (water and hydrogen) system modeling the hydrogen displacement in a storage site for radioactive waste. We suppose that the water is only in the liquid phase and is incompressible. The hydrogen in the gas phase is supposed compressible and could be dissolved into the water with the Henry law. The flow is described by the conservation of the mass of each components. The model is treated without simplified assumptions on the gas density. This model is degenerated due to vanishing terms. We establish an existence result for the nonlinear degenerate parabolic system based on new energy estimate on pressures.

1. **Introduction.** An important quantity of hydrogen can be produced by corrosion of ferrous materials in a storage site for radioactive waste. A direct consequence of this production is the growth of hydrogen pressure around alveolus. This increasing gas pressure could break the surrounding host rock and fractures could appear in the confinement materials. This problem renews the mathematical interest in the equation describing multiphase/multicomponent flows through porous media. The cases of immiscible and incompressible flows have been treated with "global pressure" introduced by G. Chavent and J. Jaffre [8] by many authors, we refer

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for instance to [15, 9, 10, 14] where existence results are obtained under various assumptions on physical data. For two immiscible compressible flows without exchange between the phase, we refer to [16, 17] where the authors obtain the existence of solution when the densities depend on the global pressure and to [19, 20] for the general case where the density of each phase depends on its own pressure. This approach is also used in [2, 3] to treat a homogenization problem of immiscible compressible water-gas flow in porous media. For miscible and compressible flow, we refer to [11, 12] for more details.

In [6], the authors derive a compositional model of compressible multiphase flow in porous media. They focus their study on models where the fluid is a mixture of two components: water (mostly liquid) and hydrogen (mostly gas). An existence result has been shown in [22] for this model under the assumptions of non degeneracy and of strictly positive saturation.

Recently in [7], the authors studied a new model of two compressible and partially miscible phase flow in porous media, applied to gas migration in an underground nuclear waste repository in the case where the velocity of the mass exchange between dissolved hydrogen and hydrogen in the gas phase is supposed finite.

Let us state the physical model used in this paper. We consider herein a porous medium saturated with a fluid composed of two phases (liquid and gas) and a mixture of two components (water and hydrogen) studied in [21]. As reported in [21], the author establish the existence of a weak solution, under non degeneracy and slow oscillation assumptions on the diagonal coefficients and with small data for the hydrogen. Our aim is to show global solution for the degenerate system without restriction on data.

The water is supposed only present in the liquid phase (no vapor of water due to evaporation). In order to define the physical model, let $T > 0$ be the final time fixed, let be Ω a bounded open subset of $\mathcal{R}^d$ $(d \geq 1)$ where $\partial\Omega$ is $\mathcal{C}^1$ and we set $Q_T = (0,T)\times\Omega$. In the sequel we consider the variables $t \in (0,T)$ and $x \in \Omega$. Now, we write the *mass conservation* of each component in Q_T

$$\begin{cases} \partial_t(\Phi s_l \rho_l^h + \Phi s_g \rho_g^h) + \operatorname{div}(\rho_l^h \mathbf{V}_l + \rho_g^h \mathbf{V}_g) - \operatorname{div}(\rho_l D_l^h \nabla X_l^h) = r_g, & (1.1) \\ \partial_t(\Phi s_l \rho_l^w) + \operatorname{div}(\rho_l^w \mathbf{V}_l) = r_w. & (1.2) \end{cases}$$

Here the subscript l and g represent respectively the liquid phase and the gas phase. Quantities $\Phi(x)$, $s_\alpha(t,x)$, $p_\alpha(t,x)$, $\rho_l^h(p_g)$, $\rho_g^h(p_g)$, $\rho_\alpha = \rho_\alpha^h + \rho_\alpha^w$, $X_l^h = \rho_l^h/\rho_l$ $\big(X_l^h + X_l^w = 1\big)$ and $D_l^h(x, s_l)$ represent respectively the (given) porosity of the medium, the saturation of the α phase $(\alpha = l, g)$, the pressure of the α phase, the density of dissolved hydrogen, the density of the hydrogen in the gas phase, the density of the α phase, the mass fraction of the hydrogen in the liquid phase, the diffusion-dispersion tensor of the hydrogen in the liquid phase. By definition of saturations, one gets

$$s_l + s_g = 1. \tag{1.3}$$

The velocity of each fluid $\mathbf{V}_\alpha$ is given by the Darcy law

$$\mathbf{V}_\alpha = -\mathbf{K}\frac{k_{r_\alpha}(s_\alpha)}{\mu_\alpha}\left(\nabla p_\alpha - \rho_\alpha(p_\alpha)\mathbf{g}\right), \tag{1.4}$$

where $\mathbf{K}(x)$ is the intrinsic (given) permeability tensor of the porous medium, k_{r_α} the relative permeability of the α phase, μ_α the constant α-phase's viscosity, p_α the α-phase's pressure and $\mathbf{g}$ the gravity. For detailed presentation of the model we refer to the presentation of the benchmark Couplex-Gaz [23] and [6, 22]. To define the hydrogen densities, we use the ideal gas law and the Henry law

$$\rho_g^h = \frac{M^h}{RT} p_g, \ \rho_l^h = M^h H^h p_g, \tag{1.5}$$

where the quantities M^h, H^h, R and T represent respectively the molar mass of hydrogen, the Henry constant for hydrogen, the universal constant of perfect gases and T the temperature.

The system (1.1)–(1.2) is not complete, since the unknowns are the two saturations and the two pressures and we have three equations namely (1.1), (1.2) and (1.3). To close the system, we introduce the capillary pressure law which links the jump of pressure of the two phases to the saturation

$$p_c(s_l) = p_g - p_l, \tag{1.6}$$

the application $s_l \mapsto p_c(s_l)$ is decreasing, $(\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) < 0$, for all $s_l \in [0,1])$ and $p_c(1) = 0$.

By these formulation, the system (1.1)–(1.2) is well defined and we choose the liquid and gas pressures as unknowns. From (1.5), the Henry law combined to the ideal gas law, to obtain that the density of hydrogen gas is proportional to the density of hydrogen dissolved

$$\rho_g^h = \mathcal{C}_1 \rho_l^h \text{ where } \mathcal{C}_1 = \frac{1}{H_h RT} (= 52.51). \tag{1.7}$$

Remark that the density of water ρ_l^w in the liquid phase is constant and from the Henry law, we can write

$$\rho_l \nabla X_l^h = \mathcal{C}_2 X_l^w \nabla p_g,$$

where $\mathcal{C}_2$ is a constant equal to $H^h M^h$.

Then the system (1.1)–(1.2) can be written as

$$(\mathfrak{P}) \begin{cases} \partial_t \left(\Phi m(s_l) \rho_l^h(p_g) \right) + \operatorname{div} \left(\rho_l^h(p_g) \mathbf{V}_l + \mathcal{C}_1 \rho_l^h(p_g) \mathbf{V}_g \right) \\ \qquad\qquad - \operatorname{div} \left(\mathcal{C}_2 X_l^w D_l^h(s_l) \nabla p_g \right) = r_g, & (1.8) \\ \partial_t \left(\Phi s_l \right) + \operatorname{div} \left(\mathbf{V}_l \right) = \dfrac{r_w}{\rho_l^w}. & (1.9) \end{cases}$$

where

$$m(s_l) = s_l + \mathcal{C}_1 s_g. \tag{1.10}$$

Note that the mass exchange between dissolved hydrogen and hydrogen in the gas phase is static using the Henry law opposite then supposed in [7].

2. Assumptions and main result.

The main point is to handle *a priori* estimates on the approximate solution. Due to the degeneracy for dissipative terms $\operatorname{div}(\rho_l^h M_\alpha \nabla p_\alpha)$, we can't control the discrete gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing. So, we are going to use the feature of global pressure to obtain uniform estimates on the gradient of the global pressure and the gradient of a capillary term to treat the degeneracy of the dissipative terms. Let summarize some useful notations in the sequel. We recall the concept of the global pressure as describe in [8]

$$M(s_l) \nabla p(t,x) = M_l(s_l) \nabla p_l(t,x) + M_g(s_g) \nabla p_g(t,x), \quad (t,x) \in Q_T \tag{2.1}$$

with the α-phase's mobility M_α and the total mobility are defined by

$$M_\alpha(s_\alpha) = k_{r_\alpha}(s_\alpha)/\mu_\alpha, \quad M(s_l) = M_l(s_l) + M_g(s_g).$$

This pressure p can be written as

$$p(t,x) = p_g(t,x) + \tilde{p}(s_l(t,x)) = p_l(t,x) + \bar{p}(s_l(t,x)), \quad (t,x) \in Q_T \tag{2.2}$$

with

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}s_l}(s_l) = -\frac{M_l(s_l)}{M(s_l)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) \text{ and } \frac{\mathrm{d}\bar{p}}{\mathrm{d}s_l}(s_l) = \frac{M_g(s_g)}{M(s_l)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l). \tag{2.3}$$

We also define the contribution of capillary terms by

$$\gamma(s_l) = -\frac{M_l(s_l)M_g(s_g)}{M(s_l)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) \geq 0 \text{ and } \mathcal{B}(s_l) = \int_0^{s_l} \gamma(z)\mathrm{d}z.$$

using these notations, we have

$$M_l(s_l)\nabla p_l = M_l(s_l)\nabla p + \nabla \mathcal{B}(s_l), \quad M_g(s_g)\nabla p_g = M_g(s_l)\nabla p - \nabla\mathcal{B}(s_l). \tag{2.4}$$

We complete the description of the model (1.8)-(1.9) by introducing boundary conditions and initial conditions. We set $\Sigma_T = (0,T)\times\partial\Omega$ and we note Γ_l the part of the boundary of Ω where the liquid saturation is imposed to one and $\Gamma_n = \Gamma\backslash\Gamma_l$. The chosen mixed boundary conditions on the pressures are

$$\begin{cases} p_g(t,x) = p_l(t,x) = 0 \text{ on } (0,T)\times\Gamma_l, \\ \mathbf{V}_l\cdot\mathbf{n} = \mathbf{V}_g\cdot\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_n, \\ X_l^w D_l^h \nabla p_g\cdot\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_n, \end{cases} \tag{2.5}$$

where $\mathbf{n}$ is the outward normal to Γ_n.

The initial conditions are defined on pressures

$$p_\alpha(t=0) = p_\alpha^0 \text{ in } \Omega, \text{ for } \alpha = l,g. \tag{2.6}$$

Next we introduce a classically physically relevant assumptions on the coefficients of the system.

(H1) The porosity $\phi \in W^{1,\infty}(\Omega)$ and there is two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \leq \phi(x) \leq \phi_1$ almost everywhere $x\in\Omega$.

(H2) There exists two positive constants k_0 and k_∞ such that

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty \text{ and } \langle \mathbf{K}(x)\xi,\xi\rangle \geq k_0|\xi|^2, \forall\xi\in\mathcal{R}^d.$$

(H3) The functions M_l and $M_g \in \mathcal{C}^0([0,1],\mathcal{R}^+)$, $M_\alpha(s_\alpha = 0) = 0$ and there is a positive constant $m_0 > 0$ such that for all $s_l \in [0,1]$,

$$M_l(s_l) + M_g(s_g) \geq m_0.$$

(H4) The density ρ_l^h is in $\mathcal{C}^1(\mathcal{R})$, increasing and there exists two positive constants $\rho_m > 0$ and $\rho_M > 0$ such that

$$0 < \rho_m \leq \rho_l^h(p_g) \leq \rho_M.$$

(H5) The capillary pressure fonction $p_c \in \mathcal{C}^1([0,1];\mathcal{R}^+)$ and there exists $\underline{p_c} > 0$ such that $\frac{\mathrm{d}p_c}{\mathrm{d}s_l} \leq -\underline{p_c} < 0$.

(H6) The functions $r_\omega,\ r_g \in L^2(Q_T)$ and $r_\omega,\ r_g \geq 0$ a.e. for all $(t,x)\in Q_T$.

(H7) The diffusion-dispersion tensor D_l^h (function of x and s_l) is a nonlinear continuous function of the liquid saturation s_l and is bounded for $x\in\Omega$ and $s_l \in [0,1]$. In addition, there exist a constant $d^* > 0$ such that $\forall v\in\mathcal{R}^d$, $\forall x\in\Omega$, $\forall s_l \in [0,1]$, $\left\langle D_l^h(x,s_l)v,v\right\rangle \geq d^*\|v\|^2$.

(H8) The function $\gamma \in C^1([0,1];\mathcal{R}^+)$ satisfies $\gamma(s_l) > 0$ for $0 < s_l < 1$ and $\gamma(0) = \gamma(1) = 0$. We assume that $\mathcal{B}^{-1}$ (the inverse of $\mathcal{B}(s_l) = \int_0^{s_l} \gamma(z)\mathrm{d}z$) is an Hölder[1] function of order θ, with $0 < \theta \le 1$, on $[0, \mathcal{B}(1)]$.

The assumptions (*H3*) and (*H5*) ensure that the function $\tilde{p}$ and $\bar{p}$ are $\mathcal{C}^1([0,1];\mathcal{R}^+)^1)$. Assumption (*H4*) ensures that the densities ρ_l and ρ_g are $\mathcal{C}^1(\mathcal{R})$ and bounded. This assumption don't cover the Henry law in the general case since $\rho_m > 0$. In fact, the assumption (*H4*) truncates the density of dissolved hydrogen to be bounded and positive. The general cases of the ideal gas and the Henry law remain an open problem since the kind of degeneracy (vanishing the gas pressure) combined with the degeneracy due to the saturation complicate considerably the analysis of the system. At our knowledge, the case of ideal gas law, when the densities are linear with respect to pressures, remains an open problem for the two compressible immiscible fluids.

Let us define the following Sobolev space

$$H^1_{\Gamma_l}(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_l\}$$

this is an Hilbert space with the norm $\|u\|_{H^1_{\Gamma_l}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.

For the system (1.1)–(1.2), we shall show only that the liquid saturation is nonnegative and consequently the functions M_l, M_g, and $\mathcal{B}$ are extended on $\mathcal{R}$ by continuous constant functions outside $[0,1]$ and then are bounded on $\mathcal{R}$. For that, we denote :

$$Z(s) = \begin{cases} 0 & \text{for } s \le 0 \\ s & \text{for } s \in [0,1] \\ 1 & \text{for } s > 1, \end{cases} \tag{2.7}$$

and the extended functions can be written

$$M_l(s_l) := M_l(Z(s_l)), \quad M_g(s_l) := M_g(Z(s_l)). \tag{2.8}$$

$$\mathcal{B}(s_l) = \int_0^{s_l} \gamma(Z(s))ds \tag{2.9}$$

In the same spirit and in order to write the saturation s_l as function of the principle unknowns p_g and p_l of the system, we extend the capillary pressure function p_c, still denoted p_c, to be continuous, bounded and strict monotony outside $[0,1]$, and the liquid saturation is defined as

$$s_l = p_c^{-1}(p_g - g_l).$$

Let us state the main result of this paper

Theorem 2.1. *Let* $(H1)$-$(H8)$ *hold and let the initial conditions* (p_g^0, p_l^0) *belongs in* $L^2(\Omega) \times L^2(\Omega)$ *with* $0 \le s_l^0 \le 1$. *Then there exists a solution* (p_g, p_l) *satisfying*

$$\mathcal{B}(s_l) \in L^2(0,T;H^1(\Omega)), s_l \ge 0 \text{ a.e. in } Q_T, \tag{2.10}$$

$$p \text{ and } p_g \in L^2(0,T;H^1_{\Gamma_l}(\Omega)), \tag{2.11}$$

$$\Phi\partial_t(\rho_l^h(p_g)m(s_l)) \in L^2(0,T;(H^1_{\Gamma_l}(\Omega))'),\ \Phi\partial_t s_l \in L^2(0,T;(H^1_{\Gamma_l}(\Omega))'), \tag{2.12}$$

[1]This means that there exists a positive constant c such that for all $a, b \in [0, \mathcal{B}(1)]$, one has $|\mathcal{B}^{-1}(a) - \mathcal{B}^{-1}(b)| \le c|a-b|^\theta$.

in the sense that for all $\varphi, \psi \in C^1(0,T;H^1_{\Gamma_l}(\Omega))$ *with* $\varphi(T,.) = \psi(T,.) = 0$,

$$\left\{\begin{aligned}
&-\int_{Q_T} \Phi m(s_l)\rho_l^h(p_g)\partial_t\varphi \mathrm{d}x\mathrm{d}t - \int_\Omega \Phi m(s_l^0)\rho_l^h(p_g^0)\varphi(0,x)\mathrm{d}x \\
&\quad + \int_{Q_T} \mathbf{K}\rho_l^h(p_g)(M_l(s_l)\nabla p + \nabla\mathcal{B}(s_l) - M_l(s_l)\rho_l(p_l)\boldsymbol{g})\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \\
&\quad + \mathcal{C}_1\int_{Q_T} \mathbf{K}\rho_l^h(p_g)M_g(s_l)(\nabla p_g - \rho_g(p_g)\boldsymbol{g})\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \qquad (2.13)\\
&\quad + \int_{Q_T} \mathcal{C}_2 X_l^w D_l^h \nabla p_g\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t = \int_{Q_T} r_g\varphi \mathrm{d}x\mathrm{d}t, \\
&-\int_{Q_T} \Phi s_l\partial_t\psi \mathrm{d}x\mathrm{d}t - \int_\Omega \Phi s_l^0\psi(0,x)\mathrm{d}x \\
&\quad + \int_{Q_T} \mathbf{K}(M_l(s_l)\nabla p + \nabla\mathcal{B}(s_l) - M_l(s_l)\rho_l(p_l)\boldsymbol{g})\cdot\nabla\psi \mathrm{d}x\mathrm{d}t \qquad (2.14)\\
&= \int_{Q_T} \frac{r_\omega}{\rho_l^w}\psi \mathrm{d}x\mathrm{d}t,
\end{aligned}\right.$$

and the initial conditions are satisfied in the sense that for all $\xi \in H^1_{\Gamma_l}(\Omega)$, *the functions* $t \to \int_\Omega \Phi\rho_l^h(p_g)m(s_l)\xi\mathrm{d}x$, *and* $t \to \int_\Omega \Phi s_l\xi\mathrm{d}x$, *are in* $C^0([0,T])$. *Furthermore, we have*

$$\left(\int_\Omega \Phi\rho_l^h(p_g)m(s_l)\xi\mathrm{d}x\right)(0) = \int_\Omega \phi\rho_l^h(p_g^0)m(s_l^0)\xi\mathrm{d}x,$$
$$\left(\int_\Omega \Phi s_l\xi\mathrm{d}x\right)(0) = \int_\Omega \Phi s_l^0\xi\mathrm{d}x.$$

3. **Energy estimates.** The notion of weak solutions is very natural provided that we explain the origin of the requirements (2.10)–(2.11). In this section, we give formally estimates on the gradient of the global pressure and on the gradient of the capillary term $\mathcal{B}$. In order to obtain these estimations, we define g_g and $\mathcal{H}_g$ by

$$g_g(p_g) := \int_0^{p_g} \frac{1}{\rho_l^h(z)}\mathrm{d}z \text{ and } \mathcal{H}_g(p_g) := \rho_l^h(p_g)g_g(p_g) - p_g.$$

The function $\mathcal{H}_g$ verifies $\mathcal{H}_g'(p_g) = (\rho_l^h(p_g))'g_g(p_g)$, $\mathcal{H}_g(0) = 0$, $\mathcal{H}_g(p_g) \geq 0$ for all p_g and $\mathcal{H}_g$ is sublinear with respect to p_g since the function ρ_l^h satisfies the assumption (H4): precisely we have $|H_g(p_g)| \leq (\frac{\rho_M}{\rho_m}+1)|p_g|$. This kind of function is introduced in [18, 19, 20].

By multiplying (1.8) by $g_g(p_g)$ and (1.9) by $\mathcal{C}_1 p_l - p_g$, after integration and summation of equations, we deduce the equality

$$\begin{aligned}
&\int_\Omega \Phi\left[\partial_t\Big(m(s_l)\rho_l^h(p_g)\Big)g_g(p_g) + \partial_t s_l\Big(\mathcal{C}_1\, p_l - p_g\Big)\right]\mathrm{d}x \\
&\quad + \mathcal{C}_1\int_\Omega \mathbf{K}M_l(s_l)(\nabla p_l - \rho_l(p_l)\mathbf{g})\cdot\nabla p_l\mathrm{d}x \\
&\quad + \mathcal{C}_1\int_\Omega \mathbf{K}M_g(s_g)(\nabla p_g - \rho_g(p_g)\mathbf{g})\cdot\nabla p_g\mathrm{d}x \qquad (3.1)\\
&\quad + \int_\Omega \mathcal{C}_2 X_l^w D_l^h\nabla p_g\cdot\nabla g_g(p_g)\mathrm{d}x = \int_\Omega \Big(r_g g_g(p_g) + \frac{r_\omega}{\rho_l^w}(\mathcal{C}_1 p_l - p_g)\Big)\mathrm{d}x.
\end{aligned}$$

To treat the first term of the above equality, let

$$\begin{aligned}\mathcal{M} &= \partial_t\Big(m(s_l)\rho_l^h(p_g)\Big)g_g(p_g) + \partial_t s_l\Big(\mathcal{C}_1\, p_l - p_g\Big)\\ &= \partial_t\Big(m(s_l)\rho_l^h(p_g)g_g(p_g)\Big) + \partial_t\Big(s_l(\mathcal{C}_1 p_l - p_g)\Big) + \mathcal{C}_1 s_l \partial_t(p_g - p_l) - \mathcal{C}_1\partial_t p_g\\ &= \partial_t\Big(m(s_l)\rho_l^h(p_g)g_g(p_g)\Big) + \partial_t\Big(s_l(\mathcal{C}_1 p_l - p_g)\Big) + \mathcal{C}_1 s_l \partial_t(p_c) - \mathcal{C}_1\partial_t p_g.\end{aligned}$$

Consider $\mathcal{N}$ an antiderivative of $s_l \mapsto s_l p_c'(s_l)$. We can write $\mathcal{M}$ as $\mathcal{M} = \partial_t\mathcal{E}$ where $\mathcal{E}$ is defined by

$$\begin{aligned}\mathcal{E} &= m(s_l)\rho_l^h(p_g)g_g(p_g) + s_l(\mathcal{C}_1 p_l - p_g) + \mathcal{C}_1\mathcal{N}(s_l) - \mathcal{C}_1 p_g\\ &= m(s_l)\Big(\rho_l^h(p_g)g_g(p_g) - p_g\Big) - \mathcal{C}_1 s_l p_c(s_l) + \mathcal{C}_1\mathcal{N}(s_l).\end{aligned}$$

From the definition of the functions $\mathcal{H}_g$ and $\mathcal{N}$, the expression of $\mathcal{E}$ is equivalent to

$$\mathcal{E} = m(s_l)\mathcal{H}_g(p_g) - \mathcal{C}_1\int_0^{s_l} p_c(z)\mathrm{d}z.$$

Integrate (3.1) over $(0,T)$, we deduce by using the assumptions (*H1*)-(*H8*), the positivity of $\mathcal{H}_g$ and the sub-linearity of $g_g(p_g)$, that

$$\begin{aligned}&\int_{Q_T} M_l|\nabla p_l|^2\mathrm{d}x\mathrm{d}t + \int_{Q_T} M_g|\nabla p_g|^2\mathrm{d}x\mathrm{d}t\\ &\quad + \int_{Q_T}\mathcal{C}_2 X_l^w D_l^h\nabla p_g\cdot\nabla g_g(p_g)\mathrm{d}x\mathrm{d}t \le C\left(1 + \|p_l\|_{L^2(Q_T)} + \|p_g\|_{L^2(Q_T)}\right),\end{aligned} \tag{3.2}$$

where $C>0$ is constant. In term of global pressure, from the relation (2.1), we have the fundamental equality

$$M_l(s_l)|\nabla p_l|^2 + M_g(s_l)|\nabla p_g|^2 = M(s_l)|\nabla p|^2 + \frac{M_l(s_l)M_g(s_l)}{M(s_l)}|\nabla p_c(s_l)|^2, \tag{3.3}$$

The relation (2.2) between the global pressure and the pressure of each phase proves the following inequality

$$\begin{aligned}\|p_l\|_{L^2(Q_T)} + \|p_g\|_{L^2(Q_T)} &\le 2\|p\|_{L^2(Q_T)} + \|\bar{p}\|_{L^2(Q_T)} + \|\tilde{p}\|_{L^2(Q_T)}\\ &\le C\|\nabla p\|_{L^2(Q_T)} + \|\bar{p}\|_{L^2(Q_T)} + \|\tilde{p}\|_{L^2(Q_T)}.\end{aligned}$$

The above inequality and the equality (3.3) combined to the estimate (3.2) ensures that $p,\ p_g \in L^2(0,T;H^1_{\Gamma_l}(\Omega))$ and $\mathcal{B}(s_l) \in L^2(0,T;H^1(\Omega))$.

4. **Construction of a regularized system.** Before establishing Theorem 2.1, we introduce the existence of regularized solutions to system (1.1)–(1.2). First we are interested in a non-degenerate system by adding a dissipative term on saturation preserving the positivity of the liquid saturation. Precisely, we consider the non-degenerate system:

$$\mathfrak{P}_\eta\begin{cases}\partial_t\left(\Phi m(s_l^\eta)\rho_l^h(p_g^\eta)\right) - \mathrm{div}(\mathbf{K}\rho_l^h(p_g^\eta)M_l(s_l^\eta)\left(\nabla p_l^\eta - \rho_l(p_l^\eta)\mathbf{g}\right))\\ -\mathcal{C}_1\mathrm{div}\left(\mathbf{K}\rho_l^h(p_g^\eta)M_g(s_g^\eta)\left(\nabla p_g^\eta - \rho_g(p_g)\mathbf{g}\right)\right) - \mathrm{div}(\mathcal{C}_2(X_l^w)^\eta(D_l^h)^\eta\nabla p_g)\\ +(\mathcal{C}_1 - 1)\eta\ \mathrm{div}(\rho_l^h(p_g)\nabla(p_g^\eta - p_l^\eta)) = r_g,\\ \partial_t\left(\Phi s_l^\eta\right) - \mathrm{div}\left(\mathbf{K}M_l(s_l^\eta)\left(\nabla p_l^\eta - \rho_l(p_l^\eta)\mathbf{g}\right)\right) - \eta\ \mathrm{div}(\nabla(p_g^\eta - p_l^\eta)) = \dfrac{r_w}{\rho_l^w},\end{cases}$$

completed with the initial conditions (2.6), and the following mixed boundary conditions,

$$\begin{cases} p_g^\eta(t,x) = p_l^\eta(t,x) = 0 & \text{on } (0,T)\times\Gamma_l \\ \big(\mathbf{V}_l^\eta + \mathcal{C}_1\mathbf{V}_g^\eta + \mathcal{C}_2 X_l^w D_l^h \nabla p_g + (\mathcal{C}_1-1)\eta\nabla(p_g^\eta - p_l^\eta)\big)\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_n \\ \big(\mathbf{V}_l^\eta - \eta\nabla(p_g^\eta - p_l^\eta)\big)\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_n \end{cases} \tag{4.1}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_n and $\mathbf{V}_\alpha^\eta = -\mathbf{K}M_\alpha(s_\alpha^\eta)(\nabla p_\alpha^\eta - \rho_\alpha(p_\alpha^\eta)\mathbf{g})$.

We state the existence of solutions of the above system $(\mathfrak{P}_\eta)$.

Theorem 4.1. *Let (H1)–(H8) hold. Let $p_g^0, p_l^0 \in L^2(\Omega)$, $0 \le s_l^0(x) \le 1$. Then, for all $\eta > 0$, there exists (p_g^η, p_l^η) satisfying*

$$\begin{aligned} &p_\alpha^\eta \in L^2(0,T;H^1_{\Gamma_l}(\Omega)),\ \Phi\partial_t(\rho_l^h(p_g^\eta)m(s_l^\eta)) \in L^2(0,T;(H^1_{\Gamma_l}(\Omega))'),\\ &s_l^\eta \ge 0 \text{ a.e. in } Q_T,\ s_l^\eta \in L^2(0,T;H^1(\Omega)),\ \Phi\partial_t s_l^\eta \in L^2(0,T;(H^1_{\Gamma_l}(\Omega))'),\\ &\rho_l^h(p_g^\eta)m(s_l^\eta) \in C^0\left([0,T];L^2(\Omega)\right), s_l^\eta \in C^0\left([0,T];L^2(\Omega)\right), \end{aligned}$$

such that for all $\varphi, \psi \in L^2(0,T;H^1_{\Gamma_l}(\Omega))$

$$\begin{aligned} &\left\langle \Phi\partial_t(m(s_l^\eta)\rho_l^h(p_g^\eta)), \varphi\right\rangle + \int_{Q_T}\mathbf{K}\rho_l^h(p_g^\eta)M_l(s_l^\eta)\left(\nabla p_l^\eta - \rho_l(p_l^\eta)\boldsymbol{g}\right)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t\\ &\quad + \mathcal{C}_1\int_{Q_T}\mathbf{K}\rho_l^h(p_g^\eta)M_g(s_l^\eta)\left(\nabla p_g^\eta - \rho_g(p_g^\eta)\boldsymbol{g}\right)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\ &\quad + \int_{Q_T}\mathcal{C}_2(X_l^w)^\eta(D_l^h)^\eta\nabla p_g^\eta\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t + (\mathcal{C}_1-1)\eta\int_{Q_T}\rho_l^h(p_g^\eta)\nabla(p_g^\eta - p_l^\eta)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\ &\qquad\qquad = \int_{Q_T} r_g\varphi\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \langle\Phi\partial_t s_l^\eta, \psi\rangle &+ \int_{Q_T}\mathbf{K}M_l(s_l^\eta)\left(\nabla p_l^\eta - \rho_l(p_l^\eta)\boldsymbol{g}\right)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\ &- \eta\int_{Q_T}\nabla(p_g^\eta - p_l^\eta)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t = \int_{Q_T}\frac{r_\omega}{\rho_l^w}\psi\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.3}$$

where the bracket $\langle .,.\rangle$ is the duality product between $L^2(0,T;(H^1_{\Gamma_l}(\Omega))')$ and $L^2(0,T;H^1_{\Gamma_l}(\Omega))$.

For the sake of clarity, we omit the index η in the problem $(\mathfrak{P}_\eta)$. The sequel of this section is devoted to the proof of Theorem 4.1. The existence of solution of non-degenerated model $(\mathfrak{P}_\eta)$ is splitted in three steps. The first one based on approached solutions solving a time discrete system with non-degenerate mobilities. For that, let be $M \in \mathbb{N}^*$ a number of time step, $\delta t = T/M$ the time step and let initialize a sequence parametrized by δt with the initial condition p_α^0. Then, if we consider $(p_g^{n-1}, p_l^{n-1}) \in (L^2(\Omega))^2$ with $\rho_l^h(p_g^{n-1})m(s_l^{n-1}) \ge 0$ and $s_l^{n-1} \ge 0$ at time $t_{n-1} = (n-1)\delta t$, for $n = 1, M-1$, a standard Leray-Schauder's fixed point theorem [24] allows to define a solution (p_g^n, p_l^n) in $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$, and

$s_l^n \in L^2(\Omega)$, solution of the system

$$\begin{aligned}
&\int_\Omega \frac{m(s_l^n)\rho_l^h(p_g^n) - m(s_l^{n-1})\rho_l^h(p_g^{n-1})}{\delta t}\varphi \mathrm{d}x + \int_\Omega \mathcal{C}_2 (X_l^w)^n (D_l^h)^n \nabla p_g^n \cdot \nabla \varphi \mathrm{d}x \\
&+ \int_\Omega \mathbf{K}\rho_l^h(p_g^n) M_l(s_l^n)\left(\nabla p_l^n - \rho_l(p_l^n)\mathbf{g}\right)\cdot \nabla\varphi \mathrm{d}x \\
&\qquad + \mathcal{C}_1 \int_\Omega \mathbf{K}\rho_l^h(p_g^n) M_g(s_g^n)\left(\nabla p_g^n - \rho_g(p_g^n)\mathbf{g}\right)\cdot\nabla\varphi\mathrm{d}x \\
&\qquad\qquad + \eta(\mathcal{C}_1 - 1)\int_\Omega \rho_l^h(p_g^n)\nabla(p_g^n - p_l^n)\cdot\nabla\varphi\mathrm{d}x = \int_\Omega (r_g)^n \varphi \mathrm{d}x,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\int_\Omega \frac{s_l^n - s_l^{n-1}}{\delta t}\psi\mathrm{d}x + \int_\Omega \mathbf{K} M_l(s_l^n)\left(\nabla p_l^n - \rho_l(p_l^n)\mathbf{g}\right)\cdot\nabla\psi\mathrm{d}x \\
&\qquad\qquad - \eta\int_\Omega \nabla(p_g^n - p_l^n)\cdot\nabla\psi\mathrm{d}x = \int_\Omega \frac{(r_\omega)^n}{\rho_l^w}\psi\mathrm{d}x,
\end{aligned} \tag{4.5}$$

for all $(\varphi,\psi)\in (H^1_{\Gamma_l}(\Omega))^2$.This technique of semi-discretization method in time has been used by Alt and Luckhaus [1] for degenerate parabolic system and has been employed in [17, 19, 7] for a porous medium.

The second step is devoted to prove that the liquid saturation is positive. We consider $\psi = (s_l^n)^-$ in (4.5) and according to the extension of the mobility of each phase $(M_l(s_l^n)(s_l^n)^- = 0)$, we deduce that $s_l \geq 0$.

The third step is devoted to pass to the limit as δt goes to zero to prove the existence of a solution of the problem $(\mathfrak{P}_\eta)$. For this, we will show some uniform estimates with respect to the time step δt to obtain uniformly bounded on some quantities.

Lemma 4.2. *(Uniform estimates with respect to δt) The solution of the system (4.4)-(4.5) satisfies*

$$\begin{aligned}
&\frac{1}{\delta t}\int_\Omega \Phi\left(\, m(s_l^n)\mathcal{H}_g(p_g^n) - m(s_l^{n-1})\mathcal{H}_g(p_g^{n-1})\right)\mathrm{d}x \\
&\quad - \frac{1}{\delta t}\int_\Omega \Phi\left(\mathcal{P}_c(s_l^n) - \mathcal{P}_c(s_l^{n-1})\right)\,\mathrm{d}x \\
&\quad + \mathcal{C}_1 k_0 \int_\Omega M_l(s_l^n)\nabla p_l^n\cdot\nabla p_l^n \mathrm{d}x + \mathcal{C}_1 k_0\int_\Omega M_g(s_g^n)\nabla p_g^n\cdot\nabla p_g^n\mathrm{d}x \\
&\quad + C_1\int_\Omega \nabla p_g^n\cdot\nabla p_g^n\mathrm{d}x + \mathcal{C}_1\eta\int_\Omega |\nabla(p_g^n - p_l^n)|^2\mathrm{d}x \\
&\quad \leq C_2\left(\|r_g^n\|^2_{L^2(\Omega)} + \|r_\omega^n\|^2_{L^2(\Omega)}\right),
\end{aligned} \tag{4.6}$$

where C does not depend on δt. The functions $\mathcal{H}_g$ and $\mathcal{P}_c$ are defined by

$$\mathcal{H}_g(p_g) := \rho_l^h(p_g)g_g(p_g) - p_g,\ \mathcal{P}_c(s_l) := \int_0^{s_l} p_c(z)\mathrm{d}z \text{ and } g_g(p_g) = \int_0^{p_g}\frac{1}{\rho_l^h(z)}\mathrm{d}z.$$

Proof. So, since g_g is concave $(g_g^{''}(\cdot)\leq 0)$, we have

$$g_g(p_g^n) \leq g_g(p_g^{n-1}) + g_g^{'}(p_g^{n-1})(p_g^n - p_g^{n-1}),$$

and from the definition of $\mathcal{H}_g$, one gets

$$\begin{aligned}
&\left(\rho_l^h(p_g^n)m(s_l^n) - \rho_l^h(p_g^{n-1})m(s_l^{n-1})\right)g_g(p_g^n) + \left(s_l^n - s_l^{n-1}\right)\left(\mathcal{C}_1 p_l^n - p_g^n\right) \\
&\qquad \geq \mathcal{H}_g(p_g^n)m(s_l^n) - \mathcal{H}_g(p_g^{n-1})m(s_l^{n-1}) - \mathcal{C}_1(s_l^n - s_l^{n-1})p_c(s_l^n).
\end{aligned}$$

Using the concavity of $\mathcal{P}_c$ we have the inequality : $(s_l^n - s_l^{n-1})p_c(s_l^n) \leq \mathcal{P}_c(s_l^n) - \mathcal{P}_c(s_l^{n-1})$, and the above inequality, we obtain the following inequality

$$\begin{aligned}\left(\rho_l^h(p_g^n)m(s_l^n) - \rho_l^h(p_g^{n-1})m(s_l^{n-1})\right) g_g(p_g^n) + \left(s_l^n - s_l^{n-1}\right)\left(\mathcal{C}_1 p_l^n - p_g^n\right) \\ \geq \mathcal{H}_g(p_g^n)m(s_l^n) - \mathcal{H}_g(p_g^{n-1})m(s_l^{n-1}) - \mathcal{C}_1\mathcal{P}_c(s_l^n) + \mathcal{C}_1\mathcal{P}_c(s_l^{n-1}).\end{aligned} \tag{4.7}$$

Now, to obtain the inequality (4.6), we just have to multiply (4.4) by $g_g(p_g^n)$ and (4.5) by $(\mathcal{C}_1 p_l^n - p_g^n)$, sum this two equations and use the inequality (4.7). □

Finally, the limit as δt goes to zero is established by standard methods to prove existence of solutions of the non degenerate system.

5. **Existence of solutions of the degenerate system.** We have shown in the previous section 4, the existence of a solution (p_g^η, p_l^η) of the problem $\mathfrak{P}_\eta$. The aim of this section is to pass to the limit as η goes to the zero to prove the main result of this paper.

The first point to do this is to obtain uniform energy estimates with respect to η. The second point is devoted to obtain uniform estimates, space and time translates which provide compactness results on solution by virtue of Kolmogorov's theorem. Next, we will be able to pass to the limit as η goes to zero.

Now, we state the following two lemmas in order to establish uniform estimates with respect to η.

Lemma 5.1. *The sequences $(s_\alpha^\eta)_\eta$ and $(p^\eta)_\eta$ satisfy*

$$s_l^\eta \geq 0 \qquad \textit{almost everywhere in } Q_T, \tag{5.1}$$

$$(p^\eta)_\eta,\ (p_g^\eta)_\eta \qquad \textit{is uniformly bounded in } L^2(0,T;H^1_{\Gamma_l}(\Omega)), \tag{5.2}$$

$$(\sqrt{\eta}\ \nabla p_c(s_l^\eta))_\eta \qquad \textit{is uniformly bounded in } L^2(Q_T), \tag{5.3}$$

$$(\sqrt{M_\alpha(s_\alpha^\eta)}\ \nabla p_\alpha^\eta)_\eta \qquad \textit{is uniformly bounded in } L^2(Q_T),\ \alpha = l, g \tag{5.4}$$

$$(\mathcal{B}(s_l^\eta))_\eta \qquad \textit{is uniformly bounded in } L^2(0,T;H^1(\Omega)), \tag{5.5}$$

$$(\Phi\partial_t(\rho_l^h(p_g^\eta)m(s_l^\eta)))_\eta \qquad \textit{is uniformly bounded in } L^2(0,T;(H^1_{\Gamma_l}(\Omega))'), \tag{5.6}$$

$$(\Phi\partial_t(s_l^\eta))_\eta \qquad \textit{is uniformly bounded in } L^2(0,T;(H^1_{\Gamma_l}(\Omega))'). \tag{5.7}$$

Proof. The positivity of the saturation (5.1) is conserved through the limit process. For the next four estimates, we just have to multiply (4.2) by $g_g(p_g^\eta) = \int_0^{p_g^\eta} \frac{1}{\rho_l^h(z)\mathrm{d}z}$ and (4.3) by $\mathcal{C}_1 p_l^\eta - p_g^\eta$ and adding them. We follow the same calculation as in section 3 to provide the energy estimates (5.2)–(5.5).

For all $\varphi,\ \psi \in L^2(0,T;H^1_{\Gamma_l}(\Omega))$ and by using the formulation (4.2)–(4.3) with the relation (2.2) between the pressure of each phase and the global pressure, we deduce from the estimations (5.2) and (5.5), that

$$|\left\langle \Phi\partial_t(\rho_l^h(p_g^\eta)m(s_l^\eta)), \varphi\right\rangle| \leq C\|\varphi\|_{L^2(0,T;H^1_{\Gamma_l}(\Omega))},$$

and

$$|\langle \Phi\partial_t(s_l^\eta), \psi\rangle| \leq C\|\psi\|_{L^2(0,T;H^1_{\Gamma_l}(\Omega))},$$

where the bracket $\langle\cdot,\cdot\rangle$ represents the duality product between $L^2(0,T;(H^1_{\Gamma_l}(\Omega))')$ and $L^2(0,T;H^1_{\Gamma_l}(\Omega))$, which establishes (5.6)–(5.7) and proves Lemma 5.1. □

In the next lemma, we derive estimates on differences of space and time translates of the function $U^\eta = \rho_l^h(p_g^\eta)m(s_l^\eta)$ which imply that the sequence $(\rho_l^h(p_g^\eta)m(s_l^\eta))_\eta$ is relatively compact in $L^1(Q_T)$.

Lemma 5.2. (*Space and time translate of* U). *Under the assumptions* $(H1)-(H8)$, *the following inequalities hold :*

$$\int_{\Omega'\times(0,T)} |U^\eta(t,x+y)-U^\eta(t,x)|\mathrm{d}x\mathrm{d}t \le \omega(|y|), \tag{5.8}$$

$$\iint_{\Omega\times(0,T-\tau)} |U^\eta(t+\tau,x)-U^\eta(t,x)|\mathrm{d}x\mathrm{d}t \le \tilde{\omega}(\tau), \tag{5.9}$$

for all $y\in\mathcal{R}^3$ *and for all* $\tau\in(0,T)$; *with* $\Omega'=\{x\in\Omega, x+y\in\Omega\}$ *and the function* ω *and* $\tilde\omega$ *are continuous, independent of* η *and satisfying* $\lim_{|y|\to0}\omega(|y|)=0$ *and* $\lim_{\tau\to0}\tilde\omega(\tau)=0$.

Proof. For the space translates, we observe that

$$\begin{aligned}
&\int_{(0,T)\times\Omega'} |U^\eta(t,x+y)-U^\eta(t,x)|\mathrm{d}x\mathrm{d}t\\
&=\int_{(0,T)\times\Omega'} \Big|\Big(\rho_l^h(p_g^\eta)m(s_l^\eta)\Big)(t,x+y)-\Big(\rho_l^h(p_g^\eta)m(s_l^\eta)\Big)(t,x)\Big|\mathrm{d}x\mathrm{d}t\\
&\le \int_{(0,T)\times\Omega'} \Big|m(s_l^\eta)(t,x+y)\big(\rho_l^h(p_g^\eta(t,x+y))-\rho_l^h(p_g^\eta(t,x))\big)\Big|\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{(0,T)\times\Omega'} \Big|\rho_l^h(p_g^\eta)(t,x)(m(s_l^\eta)(t,x+y)-m(s_l^\eta)(t,x))\Big|\mathrm{d}x\mathrm{d}t\\
&\le \mathcal{E}_1+\mathcal{E}_2,
\end{aligned}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ defined as follows

$$\mathcal{E}_1=\rho_M\int_{(0,T)\times\Omega'}\Big|s_l^\eta(t,x+y)-s_l^\eta(t,x)\Big|\mathrm{d}x\mathrm{d}t, \tag{5.10}$$

$$\mathcal{E}_2=\int_{(0,T)\times\Omega'}\Big|\rho_l^h(p_g^\eta(t,x+y))-\rho_l^h(p_g^\eta(t,x))\Big|\mathrm{d}x\mathrm{d}t. \tag{5.11}$$

To handle with the space translates on saturation, we use the fact that $\mathcal{B}^{-1}$ is an Hölder function, applying the Cauchy-Schwarz inequality and from (5.5), we deduce

$$\begin{aligned}
\mathcal{E}_1&\le C\Big[\int_{(0,T)\times\Omega'}\Big|\mathcal{B}(s_l^\eta(t,x+y))-\mathcal{B}(s_l^\eta(t,x))\Big|\mathrm{d}x\mathrm{d}t\Big]^\theta\\
&\le C\Big[\int_0^T\int_{\Omega'}\Big(\int_0^1\nabla\mathcal{B}(s_l^\eta(t,x+ry)).y\mathrm{d}r\Big)\mathrm{d}x\mathrm{d}t\Big]^\theta\\
&\le C\Big[\int_0^T\int_{\Omega'}\Big(\int_0^1\big|\nabla\mathcal{B}(s_l^\eta(t,x+ry))\big|^2\mathrm{d}r\Big)^{\frac12}|y|\mathrm{d}x\mathrm{d}t\Big]^\theta\\
&\le C|y|^\theta.
\end{aligned}\tag{5.12}$$

To treat the space translates of $\mathcal{E}_2$, we use the relationship between the gas pressure and the global pressure, namely : $p_g=p-\tilde p$ defined in (2.2), then, from the estimation on the global pressure (5.2) and the estimate (5.12) we have

$$\mathcal{E}_2\le C(|y|+|y|^\theta).$$

Define $V^\eta = \Phi U^\eta$. From assumption $(H1)$ on the porosity, we deduce thee space translates on V^η. The proof of the time translates of V^η can be found in [13] for more details. □

From the previous two lemmas, we deduce the following convergences.

Lemma 5.3. *(Strong and weak convergences). Up to a subsequence the sequence* $(s_\alpha^\eta)_\eta$, $(p^\eta := p_l^\eta + \overline{p}(s_l^\eta))_\eta$ *and* $(p_\alpha^\eta)_\eta$ *verify the following convergence*

$$
\begin{aligned}
&p^\eta \longrightarrow p && \text{weakly in } L^2(0,T;H^1_{\Gamma_l}(\Omega)), && (5.13)\\
&\mathcal{B}(s_l^\eta) \longrightarrow \mathcal{B}(s_l) && \text{weakly in } L^2(0,T;H^1(\Omega)), && (5.14)\\
&p_g^\eta \longrightarrow p_g && \text{weakly in } L^2(0,T;H^1_{\Gamma_l}(\Omega)), && (5.15)\\
&s_l^\eta \longrightarrow s_l && \text{strongly in } L^2(Q_T), \text{ a.e. in } Q_T, && (5.16)\\
&s_l \geq 0 && \text{almost everywhere in } Q_T, && (5.17)\\
&p_\alpha^\eta \longrightarrow p_\alpha && \text{almost everywhere in } Q_T, && (5.18)\\
&\Phi\partial_t(U^\eta) \longrightarrow \Phi\partial_t(\rho_l^h(p_g)m(s_l)) && \text{weakly in } L^2(0,T;H^1_{\Gamma_l}(\Omega)), && (5.19)\\
&\Phi\partial_t s_l^\eta \longrightarrow \Phi\partial_t s_l && \text{weakly in } L^2(0,T;H^1_{\Gamma_l}(\Omega)), && (5.20)
\end{aligned}
$$

where $U^\eta = \rho_l^h(p_g^\eta)m(s_l^\eta)$.

Proof. The weak convergences (5.13)–(5.15) follows from the uniform estimates (5.2) and (5.5) of lemma 5.1.By the Riesz-Frechet-Kolmogorov compactness criterion, the relative compactness of V^η in $L^1(Q_T)$ is a consequence of Lemma 5.2 and then ensures the following strong convergences

$$\phi\rho_l^h(p_g^\eta)m(s_l^\eta) \longrightarrow l \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T \ ,$$

and consequently

$$\rho_l^h(p_g^\eta)m(s_l^\eta) \longrightarrow U = l/\Phi \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T \ , \tag{5.21}$$

In order to prove the convergence (5.16), we reproduce the previous Lemma 5.2 for $V^\eta = \Phi s_l^\eta$ and as an application of the Riesz-Frechet-Kolmogorov compactness criterion we establish (5.16). And from (5.16), we deduce (5.17).

The convergence (5.21) combined with (5.16), to prove that

$$p_g^\eta \longrightarrow p_g \text{ a.e. in } Q_T \ ,$$

thanks to $m(s_l^\eta) \geq \mathcal{C}_1$ and the fact that the function $p_g \mapsto \rho_l^h(p_g)$ is increasing, then the convergence (5.18) for $\alpha = g$ is established. And again as consequence of (5.16) with the capillary pressure law, we deduce (5.18) for $\alpha = l$. At last, the weak convergence (5.19) and (5.20) is a consequence of the estimate (5.6) and (5.7). □

In order to achieve the proof of Theorem 2.1, it remains to pass to the limit as η goes to zero in the formulations (4.2)–(4.3), for all smooth test functions φ and ψ

in $C^1([0,T]; H^1_{\Gamma_l}(\Omega))$ such that $\varphi(T) = \psi(T) = 0$,

$$
\begin{aligned}
-\int_{Q_T} \Phi\rho_l^h(p_g^\eta)m(s_l^\eta)\partial_t\varphi \mathrm{d}x\mathrm{d}t + \int_{Q_T} \mathcal{C}_2 X_l^w D_l^h \nabla p_g \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t \\
+ \int_{Q_T} \mathbf{K}\rho_l^h(p_g^\eta)M_l(s_l^\eta)\left(\nabla p_l^\eta - \rho_l(p_l)\mathbf{g}\right)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \\
+ \mathcal{C}_1 \int_{Q_T} \mathbf{K}\rho_l^h(p_g^\eta)M_g(s_l^\eta)\big(\nabla p_g^\eta - \rho_g(p_g)\mathbf{g}\big)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \\
+ (\mathcal{C}_1 - 1)\eta \int_{Q_T} \rho_l^h(p_g^\eta)\nabla(p_g^\eta - p_l^\eta)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \\
= \int_{Q_T} r_g\varphi \mathrm{d}x\mathrm{d}t + \int_{Q_T} \Phi\rho_l^h(p_g^0)m(s_l^0)\varphi(0,x)\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
-\int_{Q_T} \Phi s_l^\eta \partial_t\psi \mathrm{d}x\mathrm{d}t + \int_{Q_T} \mathbf{K}M_l(s_l^\eta)\left(\nabla p_l^\eta - \rho_l(p_l)\mathbf{g}\right)\cdot\nabla\psi \mathrm{d}x\mathrm{d}t \\
- \eta\int_{Q_T} \nabla(p_g^\eta - p_l^\eta)\cdot\nabla\psi \mathrm{d}x\mathrm{d}t = \int_{Q_T} \frac{r_\omega}{\rho_l^w}\psi \mathrm{d}x\mathrm{d}t + \int_{Q_T} \Phi s_l^0 \psi(0,x)\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{5.23}
$$

The first term in (5.22) and (5.23) converge due to the strong convergence of $\rho_l^h(p_g^\eta)m(s_l^\eta)$ to $\rho_l^h(p_g)m(s_l)$ in $L^2(Q_T)$ and the strong convergence of s_l^η to s_l in $L^2(Q_T)$.

The third and fourth term in (5.22) can be written as,

$$
\begin{aligned}
\int_{Q_T} \mathbf{K}M_\alpha(s_\alpha^\eta)\rho_\alpha(p_\alpha^\eta)\nabla p_\alpha^\eta\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t = \int_{Q_T} \mathbf{K}M_\alpha(s_\alpha^\eta)\rho_\alpha(p_\alpha^\eta)\nabla p^\eta\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \\
+ \int_{Q_T} \mathbf{K}\rho_\alpha(p_\alpha^\eta)\nabla\mathcal{B}(s_\alpha^\eta)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{5.24}
$$

The two terms on the right hand side of the equation (5.24) converge arguing in two steps. Firstly, the Lebsgue theorem and the convergences (5.16) and (5.18), establish

$$
\begin{aligned}
&\rho_\alpha(p_\alpha^\eta)M_\alpha(s_\alpha^\eta)\nabla\varphi \longrightarrow \rho_\alpha(p_\alpha)M_\alpha(s_\alpha)\nabla\varphi && \text{strongly in } (L^2(Q_T))^d, \\
&\rho_\alpha(p_\alpha^\eta)\nabla\varphi \longrightarrow \rho_\alpha(p_\alpha)\nabla\varphi && \text{strongly in } (L^2(Q_T))^d.
\end{aligned}
$$

Secondly, the weak convergence on global pressure (5.13) and the weak convergence (5.14) combined to the above strong convergences allow the convergence for the terms of the right hand side of (5.24) and obtain, as $\eta \to 0$,

$$
\begin{aligned}
\int_{Q_T} \mathbf{K}M_\alpha(s_\alpha^\eta)\rho_\alpha(p_\alpha^\eta)\nabla p_\alpha^\eta\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \longrightarrow \\
\int_{Q_T} \mathbf{K}M_\alpha(s_\alpha)\rho_\alpha(p_\alpha)\nabla p\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t + \int_{Q_T} \mathbf{K}\rho_\alpha(p_\alpha)\nabla\mathcal{B}(s_\alpha)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{5.25}
$$

In the same way for the second term of the equation (5.23).The fifth term of equations (5.22) can be written as

$$
\eta\int_{Q_T} \rho_l^h(p_g^\eta)\nabla(p_g^\eta - p_l^\eta)\nabla\varphi \mathrm{d}x\mathrm{d}t = \sqrt{\eta}\int_{Q_T} \rho_l^h(p_g^\eta)(\sqrt{\eta}\nabla p_c(s_l^\eta))\nabla\varphi \mathrm{d}x\mathrm{d}t,
$$

the Cauchy-Schwarz inequality and the uniform estimate (5.3) ensure the convergence of this term to zero as η goes to zero. In the same way for the third term of equation (5.23). The other terms converge using (5.16)–(5.18) and the Lebesgue dominated convergence theorem.

The weak formulations (2.13) and (2.14) are then established. And the main theorem 2.1 is then established.

Remark 1. In general, the source terms are considered to be L^1 function or measure in order to modelize the injection and production wells. In this case, the existence of solution remains an open problem. In general, for degenerate parabolic equations, a truncate function is used to show the existence of solutions and the uniqueness of solution is required for the renormalized solution as in ([5], [4]). Here, the test functions are nonlinear and existence of solution of the system remains an open problem.

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