A COUPLED ANISOTROPIC CHEMOTAXIS-FLUID MODEL: THE CASE OF TWO-SIDEDLY DEGENERATE DIFFUSION

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Abstract. In this article, the mathematical analysis of a model arising from biology consisting of diffusion, chemotaxis with volume filling effect and transport through an incompressible fluid, is studied. Motivated by numerical and modeling issues, the global-in-time existence of weak solutions to this model is investigated. The novelty with respect to other related papers lies in the presence of two-sidedly nonlinear degenerate diffusion and of anisotropic and heterogeneous diffusion tensors where we prove global existence for Chemotaxis-Navier-Stokes system in space dimensions less or equal than four and we show uniqueness of weak solutions for Chemotaxis-Stokes system in two or three space dimensions under further assumptions.

Keywords: Degenerate parabolic equation; Navier-Stokes equations; heterogeneous and anisotropic diffusion; global existence of solutions.

1. Introduction

Chemotaxis is the movement of biological individuals towards (or away from) a chemoattractant (or chemorepellent). A vital characteristic of living organisms is the ability to sense signals in the environment and adapt their movement accordingly. This behavior enables them to locate nutrients, avoid predators or find animals of the same species. A typical model describing chemotaxis are the Keller-Segel equations derived by Keller and Segel [11] which have become one of the best-studied models in mathematical biology. In nature, cells often live in a viscous fluid so that cells and chemical substrates are also transported with the fluid, and meanwhile the motion of the fluid is under the influence of gravitational forcing generated by aggregation of cells. Thus, it is interesting and important in biology to study some phenomenon of chemotaxis on the basis of the coupled cell-fluid model. In the following, we investigate a system consisting of the parabolic chemotaxis equations with general tensors coupled to Navier-Stokes equations,

$$\left\{ \begin{array}{rcl} \partial_t N - \nabla\cdot\big(S(x)a(N)\nabla N\big) + \nabla\cdot\big(S(x)\chi(N)\nabla C\big) + u\cdot\nabla N &=& f(N), \\ \partial_t C - \nabla\cdot(M(x)\nabla C) + u\cdot\nabla C &=& -k(C)N, \\ \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla P &=& -N\nabla\phi, \\ \nabla\cdot u &=& 0, \qquad\qquad t>0, x\in\Omega, \end{array}\right. \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$, $d \leq 4$ with smooth boundary $\partial\Omega$. The experimental set-up corresponds to mixed type boundary conditions. For simplicity here we use nul flux conditions for N and C and zero Dirichlet for u to reflect the no-slip boundary conditions of the flow. Therefore, this system of equations is supplemented by the following boundary conditions on $\Sigma_t = \partial\Omega\times(0,T)$,

$$S(x)a(N)\nabla N\cdot\eta = 0, M(x)\nabla C\cdot\eta = 0, u = 0, \tag{1.2}$$

where η is the exterior unit normal to $\partial\Omega$. The initial conditions on Ω are given by,

$$N(x,0) = N_0(x), C(x,0) = C_0(x), u(x,0) = u_0(x). \tag{1.3}$$

Here N, C, u and P denotes respectively the cell density, the concentration of a chemical, the velocity field and the pressure inside the incompressible fluid. Moreover, $a(N)$ denotes the density-dependent diffusion coefficient and $\chi(N)$ is usually written in the form $\chi(N) = Nh(N)$ where h is commonly referred to as the chemotactic sensitivity function. The source term f reflects the interaction between cells such as hydrodynamics interactions. Anisotropic and heterogeneous tensors are denoted by $S(x)$ and $M(x)$. The fluid is described by an incompressible Navier-Stokes equation with the viscosity ν. It couples to N and C through transport by the fluid modelled by $u \cdot \nabla N$, $u \cdot \nabla C$ and gravitational forcing modelled by $g = -N\nabla\phi$ as an external force exerted on the fluid by the cells. In fact, this external force can be produced by different physical mechanisms such as gravity, electric and magnetic forces but in our study, we are only interested in the case of gravitational force $\nabla\phi =$ "$V_b(\rho_b - \rho)g$"z exerted by a bacterium onto the fluid along the upwards unit vector z proportional to the volume of the bacterium V_b, the gravitation acceleration $g = 9,8m/s^2$, and the density of bacteria is ρ_b (bacteria are about 10% denser than water). Moreover, since the fluid is slow, we can use the Stokes equation instead of the Navier-Stokes equation. So the system looks like,

$$
\left\{
\begin{array}{rcll}
\partial_t N - \nabla \cdot \big(S(x)a(N)\nabla N\big) + \nabla \cdot \big(S(x)\chi(N)\nabla C\big) + u \cdot \nabla N & = & f(N), & \\
\partial_t C - \nabla \cdot (M(x)\nabla C) + u \cdot \nabla C & = & -k(C)N, & \\
\partial_t u - \nu\Delta u + \nabla P & = & -N\nabla\phi, & \\
\nabla \cdot u & = & 0 & t > 0,\, x \in \Omega.
\end{array}
\right.
\tag{1.4}
$$

In the models (1.1) and (1.4), the cell density N diffuses, it moves in the direction of the chemical gradient and it is transported by the fluid. In addition to that, the chemical C also diffuses, it is also transported by the fluid and it is consumed proportional to the density of cells N, where this fact is expressed by a function $k(C)$ which is a consumption rate of the chemical by the cells. In this paper, the chemical substrate can be only consumed by the cells ($\tilde{g}(N,C) = -k(C)N$). For example, the bacteria "Bacillus subtilis" swim towards higher concentration of oxygen to survive. In other cases, such as the "Dictyostelium discoideum", the chemical can be produced and consumed ($\tilde{g}(N,C) = aN - bC$ where a and b are positive constants) to form some kind of transition to a multicellular organism. The theoretical study of this paper is valid for both cases (chemotactical transport and transport towards a nutrient) even we are only considering the first one in the sequel. There are also an another possible choice of $\tilde{g}$ as a cut-off function for which many related experiments have been given in [6, 10, 24] to describe the aggregation of a part of bacteria below an interface between two fluids, while other bacteria are rendered inactive wherever the oxygen concentration has fallen below the threshold of activity.

Motivated by experiments described in [4, 5] which explain the dynamics of anisotropic chemotaxis models in a fluid at rest ($u = 0$) and interested by numerical issues related to the dynamics of these models coupled to a viscous fluid through transport and gravitational force, we investigate in this paper the coupled anisotropic chemotaxis-fluid models (1.1) and (1.4). A detailed theoretical study of global existence and uniqueness of weak solutions of these models has been established. In fact, the existence theory in suitable functional spaces and the uniqueness can present several difficulties due to the complicated cell-fluid interaction even if it only consists of chemotaxis and linear isotropic non-degenerate diffusion coupled to the fluid. Indeed, in the case of isotropic homogeneous tensors ($S(x) = M(x) = Id$), linear diffusion ($a(N) = 1$) and a concentration-dependent sensitivity ($\chi(N,C) = N\beta(C)$ where $\beta(C)$ is the chemotactic sensitivity), several authors of chemotaxis literature have recently studied the global existence in time via finite time blow-up of a weak solution for the models (1.1) and (1.4). The main tool used to prove global existence is an existing entropy inequality. In [7], the authors proved global existence for the model (1.4) for weak potential ϕ or small initial data of the concentration C. Moreover, for $\Omega = \mathbb{R}^2$ or $\mathbb{R}^3$, by changing the consumption rate ($-k(C)N$) into a production one ($N - aC$ where $a > 0$) and by considering the stationnary equation of C, the authors in [18] proved the existence of a critical initial mass M in the model (1.4), below M we have global existence and above M we have finite time blow-up. For $\Omega = \mathbb{R}^2$, the global existence in time of a weak solution for the model (1.1) is proved in [18]. In addition to that, for the case of isotropic tensors, nonlinear diffusion ($a(N) = mN^{m-1}\nabla N$) which degenerates only at one point ($u = 0$) and for the same sensitivity ($\chi(N,C) = N\beta(C)$), global existence of a weak solution for the model (1.4) is proved in [8] for $\Omega = \mathbb{R}^2$ and also proved for $\frac{4}{3} < m \leq 2$ where Ω is bounded in $\mathbb{R}^2$. Moreover, the case of $m = \frac{4}{3}$ in the whole space $\Omega = \mathbb{R}^3$ is treated also in [18]. To our knowledge, these are the only results on models related to (1.1) and (1.4).

The purpose of this paper is twofold: on the one hand, we establish the global-in-time existence of weak solutions to the models (1.1) and (1.4) in the open bounded domain Ω ($\Omega \subset \mathbb{R}^d$, $d \leq 4$), in the presence of anisotropic and heterogeneous tensors, two-sidedly nonlinear degenerate diffusion, modified chemotactic sensitivity χ and Navier-Stokes equations. On the other hand, we prove the uniqueness of weak solutions to the system (1.4) in Ω ($\Omega \subset \mathbb{R}^d$, $d = 2, 3$) under further assumptions and regularities on the initial data.

We assume at first that chemotactical sensitivity $\chi(N)$ vanishes when $N = 0$ and $N \geq N_m$. This threshold condition has a clear biological interpretation: the cells stop to accumulate at a given point of Ω after their density attains certain threshold value $N = N_m$. This interpretation is called the effect of a threshold cell density or the volume-filling effect which has been also taken into account in the modeling of chemotaxis phenomenon in [9] and [21].

A semi-discretization technique, inspired from [19], will be first used to establish the existence of a weak solution for the regularized non-degenerate chemotaxis-Navier-Stokes system. We can refer to [23] for more details concerning the existence of weak solutions to the Navier-Stokes equations by semi-discretizing in time which it is valid even for any number of space dimensions. Next, we tend the regularization parameter to zero and we use compactness arguments, as in [1], to pass to the limit and to prove the existence of a weak solution for the degenerate system (1.1) which has two points degeneracy ($N = 0$ and N_m). Furthermore, the proof of the uniqueness statement relies on a duality technique used also in [16] for classical Keller-Segel model.

This paper is structured as follows. In section 2, we summarize the statements of the mathematical problem, we formulate the concept of weak solutions to the models (1.1) and (1.4) and we state the main theorems of global existence and uniqueness of weak solutions. Sections 3, 4 and 5 are devoted to the proof of these theorems.

2. Preliminaries and Main results

Let us now state the assumptions on the data we will use in the sequel, together with the main results we obtain in this paper.

We assume that the density-dependent diffusion coefficient $a(N)$ degenerates for $N = 0$ and $N = N_m$. This means that the diffusion vanishes when N approaches values close to the threshold $N = N_m$ and also in the absence of cell-population. Upon normalization, we can assume that the threshold density is $N_m = 1$. The main assumptions are:

$$a : [0, 1] \longmapsto \mathbb{R}^+ \text{ is continuous, } a(0) = a(1) = 0 \text{ and } a(s) > 0 \text{ for } 0 < s < 1\,, \tag{2.1}$$

$$\chi : [0, 1] \longmapsto \mathbb{R} \text{ is continuous and } \chi(0) = \chi(1) = 0\,, \tag{2.2}$$

A standard example for χ is $\chi(N) = N(1 - N)$; $N \in [0, 1]$. The positivity of χ means that the chemical attracts the cells; the repellent case is the one of negative χ. Next, we require

$$k : [0, +\infty[\longmapsto \mathbb{R}^+ \text{ is a } C^1\text{-function with } k(0) = 0 \text{ and } k'(c) > 0 \text{ for all } c \in \mathbb{R}^+\,, \tag{2.3}$$

$$f : [0, 1] \longmapsto \mathbb{R}^+ \text{ is a continuous function with } f(0) = 0\,, \tag{2.4}$$

$$\nabla\phi \in (L^\infty(\Omega))^d \text{ and } \phi \text{ is independent of time.} \tag{2.5}$$

The permeabilities S, M: $\Omega \longrightarrow \mathcal{M}_d(\mathbb{R})$ where $\mathcal{M}_d(\mathbb{R})$ is the set of symmetric matrices $d \times d$, verify

$$S_{i,j} \in L^\infty(\Omega),\ M_{i,j} \in L^\infty(\Omega),\ \forall i, j \in \{1, .., d\}\,, \tag{2.6}$$

and there exist $c_S \in \mathbb{R}_+^*$ and $c_M \in \mathbb{R}_+^*$ such that a.e. $x \in \Omega, \forall\xi \in \mathbb{R}^d$,

$$S(x)\xi \cdot \xi \geq c_S|\xi|^2,\ M(x)\xi \cdot \xi \geq c_M|\xi|^2\,. \tag{2.7}$$

Furthermore, due to (2.2), (2.1) and (2.6), one deduces the existence of $\bar{D}$ and $\bar{D_1} \in \mathbb{R}_+^*$ such that a.e. $x \in \Omega$, $\forall N \in [0, 1]$,

$$||D(x, N)||_{\mathcal{M}_d(\mathbb{R})} = ||S(x)a(N)||_{\mathcal{M}_d(\mathbb{R})} \leq \bar{D} \text{ and } ||D_1(x, N)||_{\mathcal{M}_d(\mathbb{R})} = ||S(x)\chi(N)||_{\mathcal{M}_d(\mathbb{R})} \leq \bar{D_1}\,. \tag{2.8}$$

Next, for p in $(0, \infty)$ and an integer $m \geq 0$, we denote by $W^{m,p}(\Omega)$ $\big($resp. $W_p^{2m,m}(\Omega \times [0, T])\big)$ the Banach space consisting of all elements of $L^p(\Omega)$ $\big($resp. of $L^p(\Omega \times [0, T])\big)$ having generalized derivatives up to order m (resp. derivatives of the form $\partial_t^r\partial_x^s$ with $2r + |s| \leq 2m$) inclusively that are $p-th$ power summable on Ω.

Finally, we introduce basic spaces in the study of the Navier-Stokes equation,

$$\wp = \{u \in \mathcal{D}(\Omega), \nabla \cdot u = 0\},\ V = \bar{\wp}^{H_0^1(\Omega)} \text{ and } H = \bar{\wp}^{L^2(\Omega)}, \tag{2.9}$$

where V and H are the closure of $\wp$ in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively.

To prove the uniqueness of global weak solutions to the Chemotaxis-Stokes model (1.4), we will need later the following set of additional assumptions:

$$\begin{cases} i)\, d \leq 3,\ N_0 \in L^\infty(\Omega),\ u_0 \in W^{2-\frac{2}{p},p}(\Omega),\ C_0 \in W^{2-\frac{2}{p},p}(\Omega) \text{ with sufficient } p > d, \\ ii)\, \nabla\phi \in W^{1,\infty}(\Omega),\ f \text{ is affine, } \chi \text{ of class } C^1 \text{ and the coefficients } S_{i,j} \in C^1(\bar{\Omega}). \end{cases} \tag{2.10}$$

Before we establish global existence, we first need a proper notion of a weak solution.

Definition 2.1. *Assume that $0 \leq N_0 \leq 1$, $C_0 \geq 0$, $C_0 \in L^\infty(\Omega)$, $u_0 \in L^2(\Omega)$ and $\nabla \cdot u_0 = 0$. A triple (N, C, u) is said to be a* **weak solution** *of* (1.1)-(1.3) *if*

$$0 \leq N(x,t) \leq 1,\ C(x,t) \geq 0 \text{ a.e. in } Q_T = \Omega \times [0,T],$$

$$N \in C_w(0,T;L^2(\Omega)),\ \partial_t N \in L^2(0,T;(H^1(\Omega))^{'}),\ A(N) := \int_0^N a(r)dr \in L^2(0,T;H^1(\Omega)),$$

$$C \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \cap C(0,T;L^2(\Omega));\ \partial_t C \in L^2(0,T;(H^1(\Omega))'),$$

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap C_w(0,T;H);\ \frac{du}{dt} \in L^1(0,T;V'),$$

and (N, C, u) satisfy

$$\int_0^T < \partial_t N, \psi_1 >_{(H^1)',H^1} dt + \iint_{Q_T} [S(x)a(N)\nabla N - S(x)\chi(N)\nabla C - Nu] \cdot \nabla\psi_1 \, dxdt = -\iint_{Q_T} f(N)\psi_1 \, dxdt, \tag{2.11}$$

$$\int_0^T < \partial_t C, \psi_2 >_{(H^1)',H^1} dt + \iint_{Q_T} [M(x)\nabla C - Cu] \cdot \nabla\psi_2 \, dxdt = \iint_{Q_T} -Nk(C)\psi_2 \, dxdt, \tag{2.12}$$

$$\int_0^T < \partial_t u, \psi >_{V',V} dt + \iint_{Q_T} \nabla u \cdot \nabla\psi \, dxdt + \iint_{Q_T} (u\cdot\nabla)u\psi \, dxdt = \iint_{Q_T} -N\nabla\phi\psi \, dxdt = \iint_{Q_T} g\psi \, dxdt, \tag{2.13}$$

for all ψ_1, $\psi_2 \in L^2(0,T;H^1(\Omega))$ and $\psi \in C_c^0(]0,T[;V)$, where $C_c^0(]0,T[;V)$ denotes the space of continuous functions with compact support and values in V and $C_w(0,T;L^2(\Omega))$ denotes the space of continuous functions with values in (a closed ball of) $L^2(\Omega)$ indowed with the weak topology.

Definition 2.2. *Due to the linear Stokes equation $\big((u\cdot\nabla)u = 0\big)$, a triple (N, C, u) is said to be a* **weak solution** *of* (1.2)-(1.4) *in the same sense of Definition 2.1 with better time regularity obtained for u. One has also that $\frac{du}{dt}$ belongs to $L^2(0,T;V')$ and consequently the component u belongs to $C(0,T;H)$.*

We state now our main results of global existence and uniqueness of weak solutions in the following theorems.

Theorem 2.3. *Assume that* (2.1) *to* (2.9) *hold true. If $0 \leq N_0 \leq 1$, $C_0 \geq 0$ a.e. in Ω, $C_0 \in L^\infty(\Omega)$, $u_0 \in H$ and $g \in L^2(0,T;V')$, then the system* (1.1) *has a global weak solution (N, C, u) in the sense of Definition 2.1.*

Theorem 2.4. *Suppose that there exists a constant $C_0 > 0$ such that*

$$(\chi(N_1) - \chi(N_2))^2 \leq C_0(N_1 - N_2)(A(N_1) - A(N_2)),\ \forall N_1, N_2 \in [0,1]. \tag{2.14}$$

Then, under the set (2.10) *of additional assumptions, the system* (1.4) *has a global unique weak solution.*

Condition (2.14) may be expressed in an another way given in the following Remark.

Remark 2.5. *If $a(N) = N(1-N)$ and $\chi(N) = (N(1-N))^\beta$ then the weak solution of the system* (1.4) *is unique provided $\beta \geq \frac{3}{2}$. Indeed, this result follows from $\big[$[16], Proposition 4$\big]$ which provides a useful sufficient condition which guarantees that* (2.14) *holds true. It amounts to check that the function*

$$N \to \chi'(N)a(N)^{-\frac{1}{2}} \text{ belongs to } L^\infty(0,1).$$

A simple computation is left to the reader.

Our goal now is to give a proof to the Theorem 2.3. A major difficulty for the analysis of our system (1.1) is the strong degeneracy of the diffusion term. To handle this difficulty, we replace the original diffusion term $a(N)$ by $a_\varepsilon(N) = a(N) + \varepsilon$ and we consider for each fixed $\varepsilon > 0$, the following non-degenerate problem

$$\left\{\begin{aligned} \partial_t N_\varepsilon - \nabla\cdot(S(x)a_\varepsilon(N_\varepsilon)\nabla N_\varepsilon) + \nabla\cdot(S(x)\chi(N_\varepsilon)\nabla C_\varepsilon) + \nabla\cdot(N_\varepsilon u_\varepsilon) = f(N_\varepsilon),\\ \partial_t C_\varepsilon - \nabla\cdot(M(x)\nabla C_\varepsilon) + \nabla\cdot(C_\varepsilon u_\varepsilon) = -N_\varepsilon k(C_\varepsilon),\\ \partial_t u_\varepsilon - \nu\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \nabla p_\varepsilon = -N_\varepsilon\nabla\phi,\\ \nabla\cdot u_\varepsilon = 0, \end{aligned}\right. \tag{2.15}$$

with the following boundary and initial conditions,

$$\begin{aligned} S(x)a(N_\varepsilon)\nabla N_\varepsilon\cdot\eta = 0,\ M(x)\nabla C_\varepsilon\cdot\eta = 0,\ u_\varepsilon = 0 \text{ sur } \partial\Omega\times(0,T)\,,\\ N_\varepsilon(x,0) = N_0(x),\ C_\varepsilon(x,0) = C_0(x),\ u_\varepsilon(x,0) = u_0(x). \end{aligned}$$

The assumption $\nabla\cdot u_\varepsilon = 0$ has permitted to replace $u_\varepsilon\cdot\nabla N_\varepsilon$ (resp. $u_\varepsilon\cdot\nabla C_\varepsilon$) by $\nabla\cdot(N_\varepsilon u_\varepsilon)$ (resp. $\nabla\cdot(C_\varepsilon u_\varepsilon)$) in the problem (2.15). In order to prove Theorem 2.3, we first need to prove the existence of weak solutions to the non-degenerate problem (2.15), in section 3, by using a semi-discretization in time. Then, we apply the Schauder fixed-point theorem for the N equation, we consider the C equation as a convection-diffusion parabolic equation and we use the same guidelines of [23] for the semi-discretisation in time to the Navier-Stokes problem. Next, Kolmogorov's compactness criterion (see [3]) will be used as a compactness argument to obtain the convergence to a weak solution of the system (2.15). Finally, in section 4, we tend the regularization parameter ε to zero to produce a weak solution of the original system (1.1) in the sense of Definition 2.1 as the limit of a sequence of such approximate solutions. Convergence is achieved by means of a priori estimates and compactness arguments.

3. Weak solution of the non-degenerate problem

In this section, a semi-discretization in time technique, studied in [19] for systems modelling the miscible displacement of radioactive elements in a heterogeneous porous domain, will be applied to prove the existence of weak solutions of the non-degenerate problem (2.15). For that, we will construct an approximate solution by semi-discretization in time and then we will pass to the limit using compactness arguments. The aim of this section is given by the following Proposition.

Proposition 3.1. *The non-degenerate problem* (2.15) *admits a weak solution* $(N_\varepsilon, C_\varepsilon, u_\varepsilon)$ *in the sense of Definition 2.1, such that* $\forall\psi_1,\psi_2\in L^2(0,T;H^1(\Omega))$ *and* $\psi\in C_c^0(0,T;V)$,

$$\int_0^T <\partial_t N_\varepsilon,\psi_1> dt + \iint_{Q_T}\Big[S(x)a_\varepsilon(N_\varepsilon)\nabla N_\varepsilon - S(x)\chi(N_\varepsilon)\nabla C_\varepsilon - N_\varepsilon u_\varepsilon\Big]\cdot\nabla\psi_1\,dxdt = \iint_{Q_T} f(N_\varepsilon)\psi_1\,dxdt\,, \tag{3.1}$$

$$\iint_{Q_T}\Big[\partial_t C_\varepsilon\psi_2 + [M(x)\nabla C_\varepsilon - C_\varepsilon u_\varepsilon]\cdot\nabla\psi_2\Big]\,dxdt = \iint_{Q_T} -N_\varepsilon k(C_\varepsilon)\psi_2\,dxdt\,, \tag{3.2}$$

$$\int_0^T <\partial_t u_\varepsilon,\psi>_{V',V} dt + \iint_{Q_T}\nabla u_\varepsilon\cdot\nabla\psi\,dxdt + \iint_{Q_T}(u_\varepsilon\cdot\nabla)u_\varepsilon\psi\,dxdt = \iint_{Q_T} -N_\varepsilon\nabla\phi\psi\,dxdt\,, \tag{3.3}$$

and from the definition of V, one has

$$\nabla\cdot u_\varepsilon = 0\,. \tag{3.4}$$

3.1. **Constructing of an approximating solution.** Let $T>0$ and $\tilde N\in\mathbb{N}^*$. We define $h = \Delta t = \frac{T}{\tilde N}$ as a constant time step. Then, we define two interpolation operators which are a main tool in the study of the convergence. Let E be a Banach space.

$\forall w = (w_0, w_1, ..., w_{\tilde N})\in E^{\tilde N+1}$, the constant interpolation operator is defined from $[0,T]$ to E by

$$\left\{\begin{aligned} \Pi^0_{\tilde N} w(0) = w_0\,,\\ \Pi^0_{\tilde N} w(t) = \textstyle\sum_{n=0}^{\tilde N-1} w_{n+1}\chi_{]nh,(n+1)h]}(t) \text{ if } 0<t\le T\,, \end{aligned}\right.$$

where $\chi_{]nh,(n+1)h]}(t)$ being the characteristic function in $]nh,(n+1)h]$. The linear interpolation operator from $[0,T]$ to E is defined by

$$\Pi^1_{\tilde N} w(t) = \sum_{n=0}^{\tilde N -1} \Big[(1+n-\frac{t}{h})w_n + (\frac{t}{h}-n)w_{n+1}\Big]\chi_{[nh,(n+1)h]}(t)\,.$$

The function $\Pi^1_{\tilde N} w$ is continuous and its derivate is given for all $t \neq nh$ by

$$\frac{d}{dt}\Big(\Pi^1_{\tilde N} w(t)\Big) = \sum_{n=0}^{\tilde N -1} \Big[\frac{1}{h}(w_{n+1}-w_n)\Big]\chi_{]nh,(n+1)h]}(t)\,.$$

Note that

$$||\Pi^0_{\tilde N} w||_{L^p(0,T;E)} = h(\sum_{n=0}^{\tilde N -1} ||w_n||^p_E)^{\frac{1}{p}}) \text{ if } 1 \leq p < \infty\,, \tag{3.5}$$

$$||\Pi^0_{\tilde N} w||_{L^\infty(0,T;E)} = ||\Pi^1_{\tilde N} w||_{L^\infty(0,T;E)} = \max_{n=1,..,\tilde N}(||w_n||_E)\,. \tag{3.6}$$

Now, we will define a family of approximate solutions by the following discretized scheme in time. We begin with

$$(N^{\tilde N}_{0,\varepsilon}, C^{\tilde N}_{0,\varepsilon}, u^{\tilde N}_{0,\varepsilon}) = (N_0, C_0, u_0) \text{ the given initial data.} \tag{3.7}$$

Then, when $(N^{\tilde N}_{0,\varepsilon}, C^{\tilde N}_{0,\varepsilon}, u^{\tilde N}_{0,\varepsilon}), ..., (N^{\tilde N}_{n,\varepsilon}, C^{\tilde N}_{n,\varepsilon}, u^{\tilde N}_{n,\varepsilon})$ are known, we define $(N^{\tilde N}_{n+1,\varepsilon}, C^{\tilde N}_{n+1,\varepsilon}, u^{\tilde N}_{n+1,\varepsilon})$ which satisfy

$$\begin{aligned}\frac{1}{h}\int_\Omega (N^{\tilde N}_{n+1,\varepsilon} - N^{\tilde N}_{n,\varepsilon})\psi_1\, dx + \int_\Omega \Big(S(x)a_\varepsilon(N^{\tilde N}_{n+1,\varepsilon})\nabla N^{\tilde N}_{n+1,\varepsilon} - S(x)\chi(N^{\tilde N}_{n+1,\varepsilon})\nabla C^{\tilde N}_{n+1,\varepsilon}\Big)\cdot \nabla\psi_1\, dx \\ - \int_\Omega N^{\tilde N}_{n+1,\varepsilon} u^{\tilde N}_{n,\varepsilon}\cdot\nabla\psi_1\, dx = \int_\Omega f(N^{\tilde N}_{n+1,\varepsilon})\psi_1\, dx\,,\end{aligned} \tag{3.8}$$

$$\frac{1}{h}\int_\Omega (C^{\tilde N}_{n+1,\varepsilon} - C^{\tilde N}_{n,\varepsilon})\psi_2\, dx + \int_\Omega M(x)\nabla C^{\tilde N}_{n+1,\varepsilon}\cdot\nabla\psi_2\, dx - \int_\Omega C^{\tilde N}_{n+1,\varepsilon}u^{\tilde N}_{n,\varepsilon}\cdot\nabla\psi_2\, dx = -\int_\Omega N^{\tilde N}_{n,\varepsilon}k(C^{\tilde N}_{n+1,\varepsilon})\psi_2\, dx\,, \tag{3.9}$$

$$\begin{aligned}\frac{1}{h}\int_\Omega (u^{\tilde N}_{n+1,\varepsilon} - u^{\tilde N}_{n,\varepsilon})\psi\, dx + \int_\Omega \nabla u^{\tilde N}_{n+1,\varepsilon}\cdot\nabla\psi\, dx + \int_\Omega (u^{\tilde N}_{n+1,\varepsilon}\cdot\nabla)u^{\tilde N}_{n+1,\varepsilon}\psi\, dx = -\int_\Omega N^{\tilde N}_{n+1,\varepsilon}\nabla\phi\psi\, dx\,, \\ \forall \psi_1,\psi_2 \in H^1(\Omega) \text{ and } \forall\psi\in V.\end{aligned} \tag{3.10}$$

3.2. **Confinement of $N^{\tilde N}_{n+1,\varepsilon}$ and $C^{\tilde N}_{n+1,\varepsilon}$.** The aim of this subsection is given by the following Proposition.

Proposition 3.2. *There exists $M>0$ such that for all $n = 0,...,\tilde N -1$,*

$$0 \leq N^{\tilde N}_{n+1,\varepsilon} \leq 1 \textit{ and } 0 \leq C^{\tilde N}_{n+1,\varepsilon} \leq M \textit{ a.e. } x\in\Omega.$$

Proof. First, we note that $N^- = max(-N,0)$ belongs to $H^1(\Omega)$ (see [12], p. 54). We now make use of an induction argument. One has $N^{\tilde N}_{0,\varepsilon} = N_0 \geq 0$. Suppose that $N^{\tilde N}_{n,\varepsilon} \geq 0$ and $N^{\tilde N}_{n+1,\varepsilon} < 0$ a.e. $x\in\Omega$ and choose $\psi_1 = -(N^{\tilde N}_{n+1,\varepsilon})^-$ in (3.8), then

$$\begin{aligned}-\frac{1}{h}\int_\Omega (N^{\tilde N}_{n+1,\varepsilon} - N^{\tilde N}_{n,\varepsilon})(N^{\tilde N}_{n+1,\varepsilon})^- dx - \int_\Omega D_\varepsilon(x, N^{\tilde N}_{n+1,\varepsilon})\nabla N^{\tilde N}_{n+1,\varepsilon}\cdot\nabla(N^{\tilde N}_{n+1,\varepsilon})^- dx \\ + \int_\Omega S(x)\chi(N^{\tilde N}_{n+1,\varepsilon})\nabla C^{\tilde N}_{n+1,\varepsilon}\cdot\nabla(N^{\tilde N}_{n+1,\varepsilon})^- dx - \int_\Omega N^{\tilde N}_{n+1,\varepsilon}u^{\tilde N}_{n,\varepsilon}\cdot\nabla(N^{\tilde N}_{n+1,\varepsilon})^- dx = -\int_\Omega f(N^{\tilde N}_{n+1,\varepsilon})(N^{\tilde N}_{n+1,\varepsilon})^- dx.\end{aligned}$$

Let us mention that the non-degeneracy of a_ε leads to the coercivity of the diffusive operator $D_\varepsilon = S(x)a_\varepsilon$. Next, one has

$$-\int_\Omega N^{\tilde N}_{n+1,\varepsilon}u^{\tilde N}_{n,\varepsilon}\cdot\nabla(N^{\tilde N}_{n+1,\varepsilon})^- dx = \frac{1}{2}\int_\Omega \Big(u^{\tilde N}_{n,\varepsilon}\cdot\nabla[(N^{\tilde N}_{n+1,\varepsilon})^-]^2\Big)\, dx = 0\,.$$

We then introduce the continuous and Lipschitz extensions $\tilde{\chi}$ and $\tilde{f}$ of χ and f on $\mathbb{R}$ such that

$$\tilde{\chi}(s) = \begin{cases} 0 & \text{if } s < 0 \\ \chi(s) & \text{if } 0 \le s \le 1 \\ 0 & \text{if } s > 1 \end{cases}, \quad \tilde{f}(s) = \begin{cases} 0 & \text{if } s < 0 \\ f(s) & \text{if } 0 \le s \le 1 \\ f(1) \ge 0 & \text{if } s > 1 \end{cases},$$

to obtain $S(x)\chi(N^{\tilde{N}}_{n+1,\varepsilon}) = 0$ and $f(N^{\tilde{N}}_{n+1,\varepsilon}) = 0$ for $N^{\tilde{N}}_{n+1,\varepsilon} < 0$. Therefore, we obtain

$$-\frac{1}{h}\int_\Omega N^{\tilde{N}}_{n+1,\varepsilon}(N^{\tilde{N}}_{n+1,\varepsilon})^- dx \le 0\,.$$

So $(N^{\tilde{N}}_{n+1,\varepsilon})^- = max(-N^{\tilde{N}}_{n+1,\varepsilon}, 0) = 0$ which is a contradiction with $N^{\tilde{N}}_{n+1,\varepsilon} < 0$. Consequently,

$$\forall n = 0, .., \tilde{N} - 1,\ N^{\tilde{N}}_{n+1,\varepsilon} \ge 0 \text{ a.e. } x \in \Omega\,. \tag{3.11}$$

In the other hand, by choosing $\psi_1 = (N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+$ in (3.8), one has

$$\frac{1}{h}\int_\Omega (N^{\tilde{N}}_{n+1,\varepsilon} - N^{\tilde{N}}_{n,\varepsilon})(N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+ dx + \int_\Omega D_\varepsilon(x, N^{\tilde{N}}_{n+1,\varepsilon})\nabla(N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+ \cdot \nabla(N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+ dx$$

$$-\int_\Omega S(x)\chi(N^{\tilde{N}}_{n+1,\varepsilon})\nabla C^{\tilde{N}}_{n+1,\varepsilon} \cdot \nabla(N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+ dx - \int_\Omega N^{\tilde{N}}_{n+1,\varepsilon} u^{\tilde{N}}_{n,\varepsilon} \cdot \nabla(N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+ dx = \int_\Omega f(N^{\tilde{N}}_{n+1,\varepsilon})(N^{\tilde{N}}_{n+1,\varepsilon} - 1)^+ dx.$$

Suppose that $N^{\tilde{N}}_{n,\varepsilon} \le 1$ and that $N^{\tilde{N}}_{n+1,\varepsilon} > 1$. By following the same guidelines, one can again achieve to a contradiction. Therefore,

$$\forall n = 0, .., \tilde{N} - 1,\ N^{\tilde{N}}_{n+1,\varepsilon} \le 1 \text{ a.e. } x \in \Omega\,. \tag{3.12}$$

Finally, one can similarly prove that

$$\forall n = 0, .., \tilde{N} - 1,\ 0 \le C^{\tilde{N}}_{n+1,\varepsilon} \le M \text{ a.e. } x \in \Omega\,. \tag{3.13}$$

□

The following Lemma contains a classical result, proved in [23], that will be used in the sequel.

Lemma 3.3. *Let us consider the following trilinear function:*

$$B : H^1_0(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

$$(u, v, w) \to B(u, v, w) = \int_\Omega (u \cdot \nabla v) w\, dx\,.$$

It satisfies the following properties:

$$1)\ \textit{If } \nabla \cdot u = 0 \textit{ then } B(u, v, v) = 0\,. \tag{3.14}$$

$$2)\, B \textit{ is continuous in the space dimension } d \le 4\,. \tag{3.15}$$

3.3. **Existence of a weak solution of the equation** (3.10)**.** We are able to prove the existence of a discrete solution $u^{\tilde{N}}_{n+1,\varepsilon}$ of the equation (3.10) obtained by semi-discretization in time of the equation (3.3), under the fact of the L^∞-uniform bound (3.12). We can see all the details of the proofs in [23]. For the sake of clarity, we will just give the headlines. In space dimension $d \le 4$, the existence of $u^{\tilde{N}}_{n+1,\varepsilon} \in V$ solution of (3.10) is given by the following Lemma proved in $\big[$[23], Lemma 4.3$\big]$.

Lemma 3.4. *For each fixed h and each $n \ge 1$, there exists at least one $u^{\tilde{N}}_{n+1,\varepsilon}$ satisfying* (3.10).

Moreover, we have the following estimates and results of convergence,

$$\big|\big|\Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon\big|\big|_{L^2(0,T;V)} \le d_1, \tag{3.16}$$

$$\forall h > 0,\ ||\tau_{-h}\Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon - \Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon||_{L^2(0,T-h;(L^2(\Omega)))} \le d_1 h\,, \tag{3.17}$$

$$\big|\big|\frac{\partial}{\partial t}(\Pi^1_{\tilde{N}} u^{\tilde{N}}_\varepsilon)\big|\big|_{L^2(0,T;(H^1(\Omega))')} \le d_1, \tag{3.18}$$

where $d_1 = ||u_0||_H^2 + \int_0^T ||g(s)||_{V'}^2\, ds$ and $g = \left(\Pi_{\tilde{N}}^0 N_\varepsilon^{\tilde{N}}\right)\nabla\phi \in L^\infty(Q_T)$. In addition to that, there exists $u_\varepsilon \in L^\infty(0,T;H)\cap L^2(0,T;V)$ such that, modulo a subsequence,

$$\Pi_{\tilde{N}}^0 u_\varepsilon^{\tilde{N}} \rightharpoonup u_\varepsilon \text{ weak * in } L^\infty(0,T;H)\,, \tag{3.19}$$

$$\Pi_{\tilde{N}}^0 u_\varepsilon^{\tilde{N}} \rightharpoonup u_\varepsilon \text{ weakly in } L^2(0,T;V)\,, \tag{3.20}$$

$$\Pi_{\tilde{N}}^0 u_\varepsilon^{\tilde{N}} \longrightarrow u_\varepsilon \text{ strongly in } L^2(0,T;H). \tag{3.21}$$

By (3.14), (3.15) and (2.7), we can obtain the existence of a unique $C_{n+1,\varepsilon}^{\tilde{N}}$ satisfying (3.9) as a straightforward consequence of the Lax-Milgram theorem. Moreover, we can easily establish the following uniform estimate

$$||C_{n+1,\varepsilon}^{\tilde{N}}||_{H^1(\Omega)} \leq C', \tag{3.22}$$

where C' is a constant independent of $\tilde{N}$. Ideed, it suffices to choose $C_{n+1,\varepsilon}^{\tilde{N}}$ as a function test in (3.9) and the estimate (3.22) is a straightforward consequence of the uniform bound of the function k, (2.7), (3.12), (3.13) and (3.14).

The Schauder fixed point theorem is the main tool to prove the existence of $N_{n+1}^{\tilde{N}}$ solution of (3.8). For this purpose, we introduce the following closed convex bounded subset of the Banach space $L^2(\Omega)$:

$$\mathcal{D} = \{N_{n+1,\varepsilon}^{\tilde{N}} \in L^2(\Omega);\, 0 \leq N_{n+1,\varepsilon}^{\tilde{N}}(x) \leq 1, \text{ for a.e. } x \in \Omega\}.$$

A direct application of the Lax-Milgram theorem, (3.11) and (3.12) allow to define the application θ : $\mathcal{D} \longrightarrow \mathcal{D}$ as $\theta(w) = N_{n+1,\varepsilon}^{\tilde{N}}$ where $N_{n+1,\varepsilon}^{\tilde{N}}$ is the unique function of $H^1(\Omega)$ such that $\forall \psi_1 \in H^1(\Omega)$,

$$\begin{aligned} \frac{1}{h}\int_\Omega (N_{n+1,\varepsilon}^{\tilde{N}} - N_{n,\varepsilon}^{\tilde{N}})\psi_1\, dx + \int_\Omega S(x)a_\varepsilon(w)\nabla N_{n+1,\varepsilon}^{\tilde{N}}\cdot\nabla\psi_1\, dx - \int_\Omega S(x)\chi(w)\nabla C_{n+1,\varepsilon}^{\tilde{N}}\cdot\nabla\psi_1\, dx \\ + \int_\Omega N_{n+1,\varepsilon}^{\tilde{N}} u_{n,\varepsilon}^{\tilde{N}}\cdot\nabla\psi_1\, dx = \int_\Omega f(N_{n+1,\varepsilon}^{\tilde{N}})\psi_1\, dx. \end{aligned} \tag{3.23}$$

Each fixed point of θ is a solution of (3.8). Let us show first that $\theta(\mathcal{D})$ is relatively compact in $L^2(\Omega)$. Choosing $\psi_1 = N_{n+1,\varepsilon}^{\tilde{N}}$ as a function test in (3.23), considering three positive real numbers a_1, a_2, a_3 and using (2.7), (2.8), (3.14) and Young inequality, we have

$$\begin{aligned} \frac{1}{h}\int_\Omega |N_{n+1,\varepsilon}^{\tilde{N}}|^2 dx + \gamma'\int_\Omega |\nabla N_{n+1,\varepsilon}^{\tilde{N}}|^2 dx \leq \bar{D1}||\nabla C_{n+1,\varepsilon}^{\tilde{N}}||_{L^2}||\nabla N_{n+1,\varepsilon}^{\tilde{N}}||_{L^2} + \frac{1}{h}||N_{n,\varepsilon}^{\tilde{N}}||_{L^2}||N_{n+1,\varepsilon}^{\tilde{N}}||_{L^2} \\ + ||f(N_{n+1,\varepsilon}^{\tilde{N}})||_{L^2}||N_{n+1,\varepsilon}^{\tilde{N}}||_{L^2} \leq \bar{D1}a_1||\nabla N_{n+1,\varepsilon}^{\tilde{N}}||_{L^2}^2 + \frac{\bar{D1}}{a_1}||\nabla C_{n+1,\varepsilon}^{\tilde{N}}||_{L^2}^2 + \frac{a_2}{h}||N_{n+1,\varepsilon}^{\tilde{N}}||_{L^2}^2 \\ + \frac{1}{a_2 h}||N_{n,\varepsilon}^{\tilde{N}}||_{L^2}^2 + a_3||N_{n+1,\varepsilon}^{\tilde{N}}||_{L^2}^2 + \frac{1}{a_3}||f(w)||_{L^2}^2\,. \end{aligned}$$

Then choose $a_1 = \frac{\gamma'}{2\bar{D1}}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{4h}$ and use (3.22), the uniform bound of f and of the function $N_{n,\varepsilon}^{\tilde{N}}$ (where $\tilde{N}, n$ are integers), the above estimate is reduced to

$$||N_{n+1,\varepsilon}^{\tilde{N}}||_{H^1(\Omega)} \leq \frac{1}{2h}\int_\Omega |N_{n+1,\varepsilon}^{\tilde{N}}|^2 dx + \frac{\gamma'}{2}\int_\Omega |\nabla N_{n+1,\varepsilon}^{\tilde{N}}|^2 dx \leq C\,, \tag{3.24}$$

where C is a constant independent of w. Thus $\theta(\mathcal{D})$ is bounded in $H^1(\Omega)$ and therefore $\theta(\mathcal{D})$ is relatively compact in $L^2(\Omega)$.

Let us show that θ is a continuous mapping. Let $(w_n)_n$ be a sequence in $\mathcal{D}$ and $w \in \mathcal{D}$ be such that $w_n \longrightarrow w$ in $L^2(\Omega)$ for $n \longrightarrow +\infty$. Setting $N_n = \theta(w_n)$. The objective is to show that $N_n \longrightarrow \theta(w)$ in $L^2(\Omega)$ as $n \longrightarrow +\infty$.
Extract first from $(w_n)_n$ a sequence $(w_{n_j})_j$ converging almost everywhere in Ω to w. Since $D_\varepsilon(x,w) = S(x)a_\varepsilon(w)$ and $D_1(x,w) = S(x)\chi(w)$ are bounded and continuous with respect to w, then the dominated convergence theorem claims that

$$D_\varepsilon(x, w_{n_j}) \longrightarrow D_\varepsilon(x,w) \text{ in } \left(L^2(\Omega)\right)^{d^2},$$

$$D_1(x,w_{n_j}) \longrightarrow D_1(x,w) \text{ in } (L^2(\Omega))^{d^2}.$$

The sequence $(N_n)_n$ is bounded in $H^1(\Omega)$ which is a Hilbert space thus there exists a subsequence $(N_{n_j})_q$ such that $N_{(n_j)_q} \rightharpoonup N$ in $H^1(\Omega)$ and $N_{(n_j)_q} \longrightarrow N$ in $L^2(\Omega)$ and a.e. in Ω as $q \longrightarrow +\infty$.
The equation (3.23) yields

$$\frac{1}{h}\int_\Omega (N_{(n_j)_q} - N^{\tilde N}_{n,\varepsilon})\psi_1\,dx + \int_\Omega D_\varepsilon(x,w_{(n_j)_q})\nabla N_{(n_j)_q}\cdot\nabla\psi_1\,dx - \int_\Omega D_1(x,w_{(n_j)_q})\nabla C^{\tilde N}_{n+1,\varepsilon}\cdot\nabla\psi_1\,dx$$
$$+\int_\Omega N_{(n_j)_q}\big(u^{\tilde N}_{n,\varepsilon}\cdot\nabla\psi_1\big)\,dx = \int_\Omega f(w_{(n_j)_q})\psi_1\,dx.$$

Passing to the limit as $q\to+\infty$ yields $N=\theta(w)$. Therefore, the subsequence $N_{(n_j)_q}$ converges to $\theta(w)=N$ in $L^2(\Omega)$ as $q\to+\infty$, and the same arguments also show that every subsequence of $(N_n)_n$ converging in $L^2(\Omega)$ has for limit $\theta(w)$. Hence the sequence $(N_n)_n$ has a unique accumulation point and since it is included in a relatively compact subset of $L^2(\Omega)$, the whole sequence $(N_n)_n$ converges to $\theta(w)$ in $L^2(\Omega)$ which proves θ is continuous. The compactness of θ is a consequence of (3.24) and of the compact injection of $H^1(\Omega)$ in $L^2(\Omega)$. Finally, the Schauder fixed point theorem allows to conclude on the existence of a fixed point $N^{\tilde N}_{n+1,\varepsilon}\in H^1(\Omega)$ for θ, which is a solution of (3.8).

3.4. **Estimates.** In the following Proposition, uniform a priori estimates on the interpolation of $N^{\tilde N}$ and $C^{\tilde N}$ with respect to $\tilde N$ are obtained.

Proposition 3.5. *There exist positive constants A' and A'' independent of $\tilde N$ such that*

$$||\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon||_{L^\infty(Q_T)} = \max_{n=1\dots N}||N^{\tilde N}_{n,\varepsilon}||_{L^\infty(\Omega)} \le 1,\ ||\Pi^0_{\tilde N}C^{\tilde N}_\varepsilon||_{L^\infty(Q_T)} = \max_{n=1\dots N}||C^{\tilde N}_{n,\varepsilon}||_{L^\infty(\Omega)} \le M\,, \tag{3.25}$$

$$||\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon||_{L^2(0,T;(H^1(\Omega)))} \le A',\ ||\Pi^0_{\tilde N}C^{\tilde N}_\varepsilon||_{L^2(0,T;(H^1(\Omega)))} \le A''\,, \tag{3.26}$$

$$\forall h'>0,\ ||\tau_{-h'}\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon - \Pi^0_{\tilde N}N^{\tilde N}_\varepsilon||_{L^2(0,T-h';(L^2(\Omega)))} \le A'h'\,, \tag{3.27}$$

$$\Big|\Big|\frac{\partial}{\partial t}(\Pi^1_{\tilde N}N^{\tilde N}_\varepsilon)\Big|\Big|_{L^2(0,T;(H^1(\Omega))')} \le A',\ \Big|\Big|\frac{\partial}{\partial t}(\Pi^1_{\tilde N}C^{\tilde N}_\varepsilon)\Big|\Big|_{L^2(0,T;(H^1(\Omega))')} \le A''\,, \tag{3.28}$$

$$||\Pi^1_{\tilde N}N^{\tilde N}_\varepsilon - \Pi^0_{\tilde N}N^{\tilde N}_\varepsilon||_{L^2(0,T;(H^1(\Omega))')} \le \frac{A'h^2}{3}\,. \tag{3.29}$$

Proof. The estimate (3.25) is a simple consequence of the Proposition 3.2. By Choosing $\psi_2 = C^{\tilde N}_{n+1,\varepsilon}$ as a function test in (3.9) and by considering (2.7), the inequality $(a-b)a \ge \frac{1}{2}\big(a^2-b^2\big)$, (3.14), Proposition 3.2 and the uniform bound of k, one obtains

$$\frac{1}{h}\int_\Omega\big[(C^{\tilde N}_{n+1,\varepsilon})^2 - (C^{\tilde N}_{n,\varepsilon})^2\big]\,dx + \mu\int_\Omega(\nabla C^{\tilde N}_{n+1,\varepsilon})^2\,dx \le C_1\,,$$

where C_1 is a constant independent of $\tilde N$. Multiplying by h and summing from $n=0$ to $n=\tilde N-1$,

$$\int_\Omega (C^{\tilde N}_{\tilde N,\varepsilon})^2\,dx + \mu||\nabla\Pi^0_{\tilde N}C^{\tilde N}_\varepsilon||^2_{L^2(0,T;L^2(\Omega))} \le C_1T + \int_\Omega (C^{\tilde N}_{0,\varepsilon})^2\,dx = A''\,.$$

Therefore the estimate (3.26) is proved by following the same steps for $\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon$ in (3.8).

Let us now prove (3.27), a time translate estimate of approximate solutions which is crucial to obtain compactness property for the sequence $\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon$ in $L^2(0,T;L^2(\Omega))$. Let be $h>0$, one has

$$I := ||\tau_{-h}\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon - \Pi^0_{\tilde N}N^{\tilde N}_\varepsilon||^2_{L^2(0,T-h;(L^2(\Omega)))}$$
$$= \int_0^{T-h}\int_\Omega\Big(\Pi^0_{\tilde N}N^{\tilde N}_\varepsilon(t+h,x) - \Pi^0_{\tilde N}N^{\tilde N}_\varepsilon(t,x)\Big)^2\,dxdt = \int_0^{T-h}A(t)\,dt$$

for almost every $t\in[0,T-h]$,

$$A(t) = \int_\Omega\big(N^{\tilde N}_{[\frac{t+h}{h}],\varepsilon}(x) - N^{\tilde N}_{[\frac{t}{h}],\varepsilon}(x)\big)^2\,dx$$

(denoting by $[x]$ the integer part of a real x) which also reads

$$A(t)=\int_\Omega \sum_{n=[\frac{t}{h}]}^{[\frac{t+h}{h}]-1}\big(N^{\tilde N}_{n+1,\varepsilon}(x)-N^{\tilde N}_{n,\varepsilon}(x)\big)\big(N^{\tilde N}_{[\frac{t+h}{h}],\varepsilon}(x)-N^{\tilde N}_{[\frac{t}{h}],\varepsilon}(x)\big)\,dx\,.$$

Denote by $n_0(t)=[\frac{t}{h}]$ and $n_1(t)=[\frac{t+h}{h}]$, choosing $\psi_1=(N^{\tilde N}_{n_1,\varepsilon}-N^{\tilde N}_{n_0,\varepsilon})$ in (3.8) and summing from n_0 to n_1-1, we have

$$\sum_{n=n_0}^{n_1-1}\frac{1}{h}\int_\Omega (N^{\tilde N}_{n+1,\varepsilon}-N^{\tilde N}_{n,\varepsilon})(N^{\tilde N}_{n_1,\varepsilon}-N^{\tilde N}_{n_0,\varepsilon})\,dx \le -\sum_{n=n_0}^{n_1-1}\Big[\int_\Omega D_\varepsilon(x,N^{\tilde N}_{n+1,\varepsilon})\nabla N^{\tilde N}_{n+1,\varepsilon}\cdot\nabla(N^{\tilde N}_{n_1,\varepsilon}-N^{\tilde N}_{n_0,\varepsilon})\,dx$$
$$+\int_\Omega S(x)\chi(N^{\tilde N}_{n+1,\varepsilon})\nabla C^{\tilde N}_{n+1,\varepsilon}\cdot\nabla(N^{\tilde N}_{n_1,\varepsilon}-N^{\tilde N}_{n_0,\varepsilon})\,dx+\int_\Omega N^{\tilde N}_{n+1,\varepsilon}u^{\tilde N}_{n,\varepsilon}\cdot\nabla(N^{\tilde N}_{n_1,\varepsilon}-N^{\tilde N}_{n_0,\varepsilon})\,dx$$
$$+\int_\Omega f(N^{\tilde N}_{n+1,\varepsilon})(N^{\tilde N}_{n_1,\varepsilon}-N^{\tilde N}_{n_0,\varepsilon})\,dx\Big].$$

One defines

$$v_{i,\varepsilon}=\int_\Omega|\nabla N^{\tilde N}_{i,\varepsilon}|^2\,dx \text{ and}$$
$$p_{j,\varepsilon}=\int_\Omega\Big[\bar D^2|\nabla N^{\tilde N}_{j,\varepsilon}|^2+\bar D_1^2|\nabla C^{\tilde N}_{j,\varepsilon}|^2+|\,u^{\tilde N}_{j,\varepsilon}|^2+|\,f(N^{\tilde N}_{j,\varepsilon})|\Big]\,dx\,.$$

The Young inequality implies that $I\le h(I_1+I_2+I_3)$, where

$$I_1=\int_0^{T-h}\Big(\sum_{n=n_0+1}^{n_1}p^{\tilde N}_{n,\varepsilon}\Big)\,dt,\ I_2=\int_0^{T-h}\Big(\sum_{n=n_0+1}^{n_1}v^{\tilde N}_{n_0,\varepsilon}\Big)dt,\ I_3=\int_0^{T-h}\Big(\sum_{n=n_0+1}^{n_1}v^{\tilde N}_{n_1,\varepsilon}\Big)\,dt\,.$$

Define $\chi_n(t,t+h)=1$ if $nh\in]t,t+h]$ and $\chi_n(t,t+h)=0$ if not. Thus, I_1 may be rewritten as

$$I_1=\int_0^{T-h}\sum_{n=1}^{\tilde N}N^{\tilde N}_{n,\varepsilon}\chi_n(t,t+h)dt=\sum_{n=1}^{\tilde N}N^{\tilde N}_{n,\varepsilon}\int_0^{T-h}\chi_n(t,t+h)\,dt\le h\sum_{n=1}^{\tilde N}N^{\tilde N}_{n,\varepsilon}\,,$$

since $\int_0^{T-h}\chi_n(t,t+h)\,dt\le h$. In a similar way, one can get $I_2\le h\sum_{n=1}^{\tilde N}C^{\tilde N}_{n,\varepsilon}$ and $I_3\le h\sum_{n=1}^{\tilde N}C^{\tilde N}_{n,\varepsilon}$. Finally, we deduce that

$$I\le\sum_{i=1}^{\tilde N}h^2(p^{\tilde N}_{i,\varepsilon}+2v^{\tilde N}_{i,\varepsilon})\,.$$

Estimate (3.26) and (2.4) lead to (3.27).

To prove estimate (3.28), remark first that

$$\Big|\Big|\frac{\partial}{\partial t}\Big(\Pi^1_{\tilde N}N^{\tilde N}_\varepsilon\Big)\Big|\Big|^2_{L^2(0,T;(H^1(\Omega))')}=\sum_{n=0}^{\tilde N-1}h\Big|\Big|\frac{\partial}{\partial t}(N^{\tilde N}_{n+1,\varepsilon})\Big|\Big|^2_{(H^1(\Omega))'}=\sum_{n=0}^{\tilde N-1}\frac{1}{h}||N^{\tilde N}_{n+1,\varepsilon}-N^{\tilde N}_{n,\varepsilon}||^2_{(H^1(\Omega))'}\,. \tag{3.30}$$

Let us choose $\psi_1\in H^1(\Omega)$ as a test function in (3.8). The Cauchy-Schwarz inequality and property (2.8) imply that

$$\Big|\frac{1}{h}<N^{\tilde N}_{n+1,\varepsilon}-N^{\tilde N}_{n,\varepsilon},\psi_1>_{(H^1(\Omega))',(H^1(\Omega))}\Big|=\Big|\frac{1}{h}\int_\Omega(N^{\tilde N}_{n+1,\varepsilon}-N^{\tilde N}_{n,\varepsilon})\psi_1\,dx\Big|$$
$$\le\bar D||\nabla N^{\tilde N}_{n+1,\varepsilon}||_{(L^2(\Omega))^d}||\nabla\psi_1||_{(L^2(\Omega))^d}+\bar D_1||\nabla C^{\tilde N}_{n+1,\varepsilon}||_{(L^2(\Omega))^d}||\nabla\psi_1||_{(L^2(\Omega))^d}$$
$$+||u^{\tilde N}_{n,\varepsilon}||_{(L^2(\Omega))^d}||\nabla\psi_1||_{(L^2(\Omega))^d}+||f(N^{\tilde N}_{n+1,\varepsilon})||_{(L^2(\Omega))^d}||\psi_1||_{(L^2(\Omega))^d}\,. \tag{3.31}$$

Simplifying by $||\psi_1||_{H^1(\Omega)}$, raising to the square and using the following inequality $(a+b)^2\le 2(a^2+b^2)$ for $a,b\ge 0$, one obtains for all n that

$$\frac{1}{h^2}||N^{\tilde N}_{n+1,\varepsilon}-N^{\tilde N}_{n,\varepsilon}||^2_{(H^1(\Omega))'}\le 2(\bar D^2||\nabla N^{\tilde N}_{n+1,\varepsilon}||^2_{(L^2(\Omega))^d}+\bar D_1^2||\nabla C^{\tilde N}_{n+1,\varepsilon}||^2_{(L^2(\Omega))^d}+||u^{\tilde N}_{n,\varepsilon}||^2_{(L^2(\Omega))^d}$$
$$+||f(N^{\tilde N}_{n+1,\varepsilon})||^2_{(L^2(\Omega))^d})\,.$$

Multiplying by h, summing from $n=0$ to $n=\tilde{N}-1$, recalling (3.5) and using estimates (3.16) and (3.26), one obtains the existence of a positive constant A' not depending on $\tilde{N}$ satisfying (3.28).

Finally, we prove (3.29). Indeed,

$$\left|\left|\Pi^1_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} - \Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon}\right|\right|_{L^2(0,T;(H^1(\Omega))')} = \sum_{n=0}^{\tilde{N}-1} \int_{nh}^{(n+1)h} \left|\left|(1+n-\frac{t}{h})\big[N^{\tilde{N}}_{n,\varepsilon} - N^{\tilde{N}}_{n+1,\varepsilon}\big]\right|\right|^2_{(H^1(\Omega))'} dt \le \frac{A'h^2}{3},$$

as a consequence of

$$\int_{nh}^{(n+1)h} (1+n-\frac{t}{h})^2\, dt = -h\int_{nh}^{(n+1)h} \frac{-1}{h}(1+n-\frac{t}{h})^2\, dt = -h\Big[\frac{1}{3}(1+n-\frac{t}{h})^3\Big]_{nh}^{(n+1)h} = \frac{h}{3}$$

and

$$\sum_{n=0}^{\tilde{N}-1} \left|\left|N^{\tilde{N}}_{n+1,\varepsilon} - N^{\tilde{N}}_{n,\varepsilon}\right|\right|^2_{(H^1(\Omega))'} \le A'h \text{ (using estimate (3.28))}.$$

□

3.5. **Passing to the limit.** As $\tilde{N}$ tends to $+\infty$, we conclude on the existence of a weak solution of the non-degenerate problem.

Proposition 3.6. *There exist subsequences of* $(\Pi^0_{\tilde{N}} C^{\tilde{N}}_{\varepsilon})_{\tilde{N}}$ *and* $(\Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon})_{\tilde{N}}$ *still denotes by* $(\Pi^0_{\tilde{N}} C^{\tilde{N}}_{\varepsilon})_{\tilde{N}}$ *and* $(\Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon})_{\tilde{N}}$, *and functions* N_ε *and* C_ε *such that*

(3.32) $$\Pi^0_{\tilde{N}} C^{\tilde{N}}_{\varepsilon} \rightharpoonup C_\varepsilon \text{ and } \Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} \rightharpoonup N_\varepsilon \text{ weakly-* in } L^\infty(Q_T),$$

(3.33) $$\Pi^0_{\tilde{N}} C^{\tilde{N}}_{\varepsilon} \rightharpoonup C_\varepsilon \text{ and } \Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} \rightharpoonup N_\varepsilon \text{ weakly in } L^2(0,T;H^1(\Omega)),$$

(3.34) $$\Pi^0_{\tilde{N}} C^{\tilde{N}}_{\varepsilon} \longrightarrow C_\varepsilon \text{ and } \Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} \longrightarrow N_\varepsilon \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,$$

(3.35) $$\frac{\partial}{\partial t}\Big(\Pi^1_{\tilde{N}} C^{\tilde{N}}_{\varepsilon}\Big) \rightharpoonup \frac{\partial C_\varepsilon}{\partial t} \text{ and } \frac{\partial}{\partial t}\Big(\Pi^1_{\tilde{N}} N^{\tilde{N}}_{\varepsilon}\Big) \rightharpoonup \frac{\partial N_\varepsilon}{\partial t} \text{ weakly in } L^2(0,T;(H^1(\Omega))'),$$

(3.36) $$\Pi^1_{\tilde{N}} C^{\tilde{N}}_{\varepsilon} \rightharpoonup C_\varepsilon \text{ and } \Pi^1_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} \rightharpoonup N_\varepsilon \text{ weakly-* in } L^\infty(Q_T),$$

as $\tilde{N} \longrightarrow +\infty$. *Moreover,* N_ε *and* C_ε *verify*

(3.37) $$N_\varepsilon(0,x) = N_{0,\varepsilon}(x) \text{ and } C_\varepsilon(0,x) = C_{0,\varepsilon}(x) \text{ a.e.},$$

(3.38) $$0 \le N_\varepsilon(t,x) \le 1 \text{ and } C_\varepsilon(t,x) \ge 0 \text{ for a.e. } (t,x) \in [0,T]\times\Omega = Q_T.$$

Proof. Each assertion of this Proposition will be proved for the N equation and by the same arguments, we can prove the same convergences related to the C equation. Assertions (3.32), (3.33) and (3.35) are straightforward applications of the estimates (3.25), (3.26) and (3.28) of Proposition 3.5. By proving (3.33) and (3.27) which are respectively space and time translate estimates of approximate solutions, the assumptions of the Kolmogorov's compactness criterion are satisfied. Therefore, we can deduce that the sequence $(\Pi^0_{\tilde{N}} N^{\tilde{N}}_{\varepsilon})_N$ is relatively compact in $L^2(0,T;L^2(\Omega))$. So, modulo a subsequence, one has (3.34).

Next, the proof of the assertion (3.36) is similar to the proof of (3.32) because we have the same L^∞-norm of the interpolation operators defined in (3.6). Moreover, estimates (3.28) and Proposition 3.2 permit to prove that there exist w and a subsequence $N^{\tilde{N}}_{\varepsilon}$ such that as $\tilde{N} \longrightarrow +\infty$,

$$\Pi^1_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} \rightharpoonup w \text{ weakly-* in } L^\infty(Q_T),$$

$$\frac{\partial}{\partial t}\Big(\Pi^1_{\tilde{N}} N^{\tilde{N}}_{\varepsilon}\Big) \rightharpoonup \frac{\partial w}{\partial t} \text{ weakly in } L^2(0,T;(H^1(\Omega))').$$

The space $L^\infty(\Omega)$ being compactly embedded into $(H^1(\Omega))'$, one deduces (see [22])

(3.39) $$\Pi^1_{\tilde{N}} N^{\tilde{N}}_{\varepsilon} \longrightarrow w \text{ strongly in } C(0,T;(H^1(\Omega))'), \text{ as } \tilde{N} \longrightarrow +\infty.$$

Estimates (3.29) and (3.34) imply that $\Pi^1_{\tilde{N}} N^{\tilde{N}}_\varepsilon \longrightarrow N_\varepsilon$ strongly in $L^2(0,T;(H^1(\Omega))')$ as $\tilde{N} \longrightarrow +\infty$. Due to the uniqueness of the limit, we obtain in (3.39)

$$w = N_\varepsilon \,. \tag{3.40}$$

The sequence $(N_\varepsilon)_\varepsilon$ belongs to the space $C(0,T;(H^1(\Omega))') \subseteq C_w(0,T;(H^1(\Omega))')$. Indeed, this last inclusion is true because: For a sequence $(t_n)_n$ converging to t in $[0,T]$, as $n \to +\infty$, one has

$$| < N_\varepsilon(t_n) - N_\varepsilon(t), v > | \leq ||N_\varepsilon(t_n) - N_\varepsilon(t)||_{(H^1(\Omega))'} ||v||_{H^1(\Omega)} \longrightarrow 0 \,.$$

As we have $L^2(\Omega) \hookrightarrow (H^1(\Omega))'$ is continuous and $N_\varepsilon \in C_w(0,T;(H^1(\Omega))') \cap L^\infty(0,T;L^2(\Omega))$ therefore $N_\varepsilon \in C_w(0,T;(L^2(\Omega)))$ (see [2], Chapter 2, Lemma 2.5.7).

Assertion (3.37) is proved using (3.39), $\Pi^1_{\tilde{N}} N^{\tilde{N}}_\varepsilon(0) \longrightarrow N_\varepsilon(0,x)$ strongly in $(H^1(\Omega))'$, and since $\forall N,\ \Pi^1_{\tilde{N}} N^{\tilde{N}}_\varepsilon(0) = N_{0,\varepsilon}(x)$ given by the definition of $\Pi^1_{\tilde{N}}$. Therefore, $N_\varepsilon(0,x) = N_{0,\varepsilon}(x)$ a.e. $x \in \Omega$.

Finally, let us prove assertion (3.38). Define $\tilde{n} = [t\frac{\tilde{N}}{T}] + 1 = [\frac{t}{h}] + 1$, $\forall t \in [0,T]$ and remark that $\Pi^0_{\tilde{N}} N^{\tilde{N}}_\varepsilon(t,x) = N^{\tilde{N}}_{\tilde{n},\varepsilon}(x)$. By noticing that $\tilde{n}h = \big([(t\frac{\tilde{N}}{T})] + 1\big)\frac{T}{\tilde{N}} \longrightarrow t$ as $\tilde{N} \longrightarrow +\infty$ and $\Pi^0_{\tilde{N}} N^{\tilde{N}}_\varepsilon(t,x) \longrightarrow N_\varepsilon(t,x)$ a.e. $(t,x) \in Q_T$, as $\tilde{N} \longrightarrow +\infty$ using (3.34) . As we have, $0 \leq N^{\tilde{N}}_{\tilde{n},\varepsilon}(x) \leq 1$ then $0 \leq N_\varepsilon(x,t) \leq 1$. Similarly to (3.35), one obtains

$$\frac{\partial}{\partial t}\Big(\Pi^1_{\tilde{N}} C^{\tilde{N}}_\varepsilon\Big) \rightharpoonup \frac{\partial C_\varepsilon}{\partial t} \text{ in } L^2(0,T;(H^1(\Omega))') \text{ as } \tilde{N} \longrightarrow +\infty \,. \tag{3.41}$$

□

It remains now to check whether the variational equalities still hold to prove that the triple $(N_\varepsilon, C_\varepsilon, u_\varepsilon)$ is a weak solution of (2.15).

The equation (3.10) can be written as, $\forall \psi \in L^2(0,T;V)$,

$$\int_0^T < \partial_t(\Pi^1_{\tilde{N}} u^{\tilde{N}}_\varepsilon), \psi > dt + \iint_{Q_T} \nabla(\Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon) \cdot \nabla\psi \, dxdt + \iint_{Q_T} (\Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon \cdot \nabla)\Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon \psi \, dxdt$$
$$= \iint_{Q_T} \Pi^0_{\tilde{N}} N^{\tilde{N}}_\varepsilon \nabla\phi \cdot \psi \, dxdt \,.$$

In other words, $P_1 + P_2 + P_3 = P_4$. As $\tilde{N}$ goes to $+\infty$, one obtains

$$P_1 \longrightarrow \int_0^T < \frac{\partial u_\varepsilon}{\partial t}, \psi > dt \text{ using (3.18)}, \ \ P_2 \longrightarrow \iint_{Q_T} \nabla u_\varepsilon . \nabla\psi \, dxdt \text{ using (3.20)},$$

$$P_3 \longrightarrow \iint_{Q_T} (u_\varepsilon \cdot \nabla) u_\varepsilon \psi \, dxdt \text{ using (3.20) and (3.21)}, \ \ P_4 \longrightarrow \iint_{Q_T} N_\varepsilon \nabla\phi \cdot \psi \, dxdt \text{ using (2.5) and (3.34)}.$$

We mention that the detailed convergence of P_3 is proved in [23].

The equation (3.9) can be written as: $\forall \psi_2 \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T < \partial_t(\Pi^1_{\tilde{N}} C^{\tilde{N}}_\varepsilon), \psi_2 > dt + \iint_{Q_T} M(x)\nabla(\Pi^0_{\tilde{N}} C^{\tilde{N}}_\varepsilon) \cdot \nabla\psi_2 \, dxdt - \iint_{Q_T} \Pi^0_{\tilde{N}} C^{\tilde{N}}_\varepsilon \big(\tau_h \Pi^0_{\tilde{N}} u^{\tilde{N}}_\varepsilon \cdot \nabla\psi_2\big) \, dxdt$$
$$= - \iint_{Q_T} (\tau_h \Pi^0_{\tilde{N}} N^{\tilde{N}}_\varepsilon) k(\Pi^0_{\tilde{N}} C^{\tilde{N}}_\varepsilon) \psi_2 \, dxdt \,.$$

For the sake of simplicity, we can rewrite this equality as $V_1 + V_2 + V_3 = -V_4$. As $\tilde{N}$ goes to $+\infty$,

$$V_1 \longrightarrow \int_0^T < \frac{\partial C_\varepsilon}{\partial t}, \psi_2 > dt \text{ using (3.41)}.$$

$$V_2 \longrightarrow \iint_{Q_T} M(x)\nabla C_\varepsilon \cdot \nabla\psi_2 \, dxdt \text{ using (3.33) and the boundedness of the tensor } M(x).$$

$$V_3 \longrightarrow - \iint_{Q_T} C_\varepsilon u_\varepsilon \cdot \nabla\psi_2 \, dxdt \,,$$

indeed, the weak convergence of $\tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon$ to u_ε in $L^2(Q_T)$ may be sufficient to prove this convergence,

$$V_3 - \Big(- \iint_{Q_T} C_\varepsilon u_\varepsilon \cdot \nabla \psi_2 \, dxdt \Big) = - \iint_{Q_T} \Pi^0_{\tilde N} C^{\tilde N}_\varepsilon (\tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon \cdot \nabla \psi_2) \, dxdt + \iint_{Q_T} C_\varepsilon u_\varepsilon \cdot \nabla \psi_2 \, dxdt$$

$$= - \iint_{Q_T} (\Pi^0_{\tilde N} C^{\tilde N}_\varepsilon - C_\varepsilon) \tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon \cdot \nabla \psi_2 \, dxdt - \iint_{Q_T} C_\varepsilon (\tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon - u_\varepsilon) \cdot \nabla \psi_2 \, dxdt \, .$$

The convergence of the first term is a consequence of the dominated convergence theorem of Lebesgue. The weak convergence of $\tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon$ to u_ε in $L^2(Q_T)$ is sufficient for the second term to converge to 0. Otherwise, we can also prove the convergence of the term V_3 by using the strong $L^2(Q_T)$-convergence of $\tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon$ to u_ε deduced from (3.17) and (3.21). Therefore,

$$\Big| V_3 - \Big(- \iint_{Q_T} u_\varepsilon \cdot \nabla C_\varepsilon \psi_2 \, dxdt \Big) \Big| \le \big|\big| (\Pi^0_{\tilde N} C^{\tilde N}_\varepsilon - C_\varepsilon) \cdot \nabla \psi_2 \big|\big|_{L^2(Q_T)} \big|\big| \tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon \big|\big|_{L^2(Q_T)}$$

$$+ \big|\big| C_\varepsilon \big|\big|_{L^\infty(Q_T)} \big|\big| \tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon - u_\varepsilon \big|\big|_{L^2(Q_T)} \big|\big| \nabla \psi_2 \big|\big|_{L^2(Q_T)} \longrightarrow 0 \, .$$

Finally, we prove that

$$V_4 \longrightarrow \iint_{Q_T} N_\varepsilon k(C_\varepsilon) \psi_2 \, dxdt \, ,$$

indeed,

$$\Big| V_4 - \iint_{Q_T} N_\varepsilon k(C_\varepsilon) \psi_2 \, dxdt \Big| \le ||\Pi^0_{\tilde N} N^{\tilde N}_\varepsilon||_{L^\infty(Q_T)} ||k(\Pi^0_{\tilde N} C^{\tilde N}_\varepsilon) - k(C_\varepsilon)||_{L^2(Q_T)} ||\psi_2||_{L^2(Q_T)}$$

$$+ ||k(C_\varepsilon)||_{L^\infty(Q_T)} ||\Pi^0_{\tilde N} N^{\tilde N}_\varepsilon - N_\varepsilon||_{L^2(Q_T)} ||\psi_2||_{L^2(Q_T)} \, .$$

Using (3.34) and the L^∞-uniform bound of k (k is a C^1-function on a compact) and $\Pi^0_{\tilde N} N^{\tilde N}_\varepsilon$, one obtains the desired convergence.

Equation (3.8) yields: $\forall \psi_1 \in L^2(0,T; H^1(\Omega))$,

$$\int_0^T < \partial_t (\Pi^1_{\tilde N} N^{\tilde N}_\varepsilon), \psi_1 > dt + \iint_{Q_T} D_\varepsilon(x, \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon) \nabla (\Pi^0_{\tilde N} N^{\tilde N}_\varepsilon) \cdot \nabla \psi_1 \, dxdt$$

$$= \iint_{Q_T} D_1(x, \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon) \nabla (\Pi^0_{\tilde N} C^{\tilde N}_\varepsilon) \cdot \nabla \psi_1 \, dxdt + \iint_{Q_T} \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon \big(\tau_h \Pi^0_{\tilde N} u^{\tilde N}_\varepsilon \cdot \nabla \psi_1 \big) \, dxdt$$

$$+ \iint_{Q_T} f(\Pi^0_{\tilde N} N^{\tilde N}_\varepsilon) \psi_1 \, dxdt \, .$$

In other words, $U_1 + U_2 = U_3 + U_4 + U_5$. As $\tilde N$ goes to $+\infty$,

$$U_1 \longrightarrow \int_0^T < \frac{\partial N_\varepsilon}{\partial t}, \psi_1 > dt \text{ using (3.35).}$$

$$U_2 \longrightarrow \iint_{Q_T} D_\varepsilon(x, N_\epsilon) \nabla N_\varepsilon \cdot \nabla \psi_1 \, dxdt \, ,$$

indeed, one has $\Pi^0_{\tilde N} N^{\tilde N}_\varepsilon \longrightarrow N_\varepsilon$ a.e. in Q_T and therefore $D_\varepsilon(x, \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon) \longrightarrow D_\varepsilon(x, N_\varepsilon)$ a.e. in Q_T because $D_\varepsilon = S(x) a_\varepsilon$ is continuous with respect to the second variable. Then, it follows from the dominated convergence of Lebesgue, (3.33) and $D_\varepsilon(x, N_\epsilon) \nabla \psi_1 \in L^2(Q_T)$ that

$$\Big| U_2 - \iint_{Q_T} D_\varepsilon(x, N_\epsilon) \nabla (N_\varepsilon) \nabla \psi_1 \, dxdt \Big| \le \big|\big| \big(D_\varepsilon(x, \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon) - D_\varepsilon(x, N_\varepsilon) \big) \nabla \psi_1 \big|\big|_{L^2(Q_T)} ||\nabla \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon||_{L^2(Q_T)}$$

$$+ \iint_{Q_T} D_\varepsilon(x, N_\epsilon) (\nabla \Pi^0_{\tilde N} N^{\tilde N}_\varepsilon - \nabla N_\varepsilon) \cdot \nabla \psi_1 \, dxdt \longrightarrow 0 \, .$$

With similar arguments, one can prove that

$$U_3 \longrightarrow \iint_{Q_T} D_1(x, N_\varepsilon) \nabla C_\varepsilon \cdot \nabla \psi_1 \, dxdt \, ,$$

by using (3.33) and the continuity of $D_1 = S(x)\chi(N_\varepsilon)$ with respect to N_ε.

$$U_4 \longrightarrow \iint_{Q_T} N_\varepsilon u_\varepsilon \cdot \nabla\psi_1 \, dxdt \text{ proved in a similar way as the convergence of } V_3.$$

$$U_5 \longrightarrow \iint_{Q_T} f(N_\varepsilon)\psi_1 \, dxdt \,,$$

indeed, the Lipschitz continuity of the function f on $[0,1]$ and (3.34) lead to this last convergence. Through the same guidelines, we obtain the convergence of (3.10) to the weak formulation of the evolution Navier-Stokes equation given in Definition 2.1 (see [23]).

4. Weak solution of the degenerate problem

The aim of this section is to send the regularization parameter ε to zero in sequences of weak solutions of problem (2.15) to obtain a weak solution of the original system (1.1) in the sense of Definition 2.1. Note that, for each fixed $\varepsilon > 0$, we have shown the existence of a solution $(N_\varepsilon, C_\varepsilon, u_\varepsilon)$ such that $0 \le N_\varepsilon(x,t) \le 1$ and $0 \le C_\varepsilon(x,t) \le M$ a.e. in Q_T. Then to conclude on the existence of a weak solution of (1.1), we shall need to prove the following uniform a priori estimates.

4.1. **Estimates.** Choosing the approximate solution C_ε as a function test ψ_2 in (3.2), using (3.14) and $N_\varepsilon(x,t) \ge 0$, one has

$$\frac{d}{dt}\int_\Omega |C_\varepsilon|^2 dx + c_M \iint_{Q_T} |\nabla C_\varepsilon|^2 \, dxdt + \iint_{Q_T} |C_\varepsilon|^2 \, dxdt \le C \,, \tag{4.1}$$

where C is a constant independent of ε. Consequently,

$$(C_\varepsilon)_\varepsilon \text{ is a bounded sequence in } L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \,. \tag{4.2}$$

Thus there exist a solution $C \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega))$ and a subsequence of C_ε still denotes as the sequence such that, as ε goes to 0,

$$C_\varepsilon \rightharpoonup C \text{ weakly-* in } L^\infty(Q_T) \,, \tag{4.3}$$

$$C_\varepsilon \rightharpoonup C \text{ weakly in } L^2(0,T;H^1(\Omega)) \,. \tag{4.4}$$

One can easily deduce from the weak formulation (3.2) applied to a test function $\varphi \in L^2(0,T;(H^1(\Omega)))$ that

$$(\frac{\partial C_\varepsilon}{\partial t})_\varepsilon \text{ is a bounded sequence in } L^2(0,T;(H^1(\Omega))') \,. \tag{4.5}$$

Therefore,

$$\frac{\partial C_\varepsilon}{\partial t} \rightharpoonup \frac{\partial C}{\partial t} \text{ weakly in } L^2(0,T;(H^1(\Omega))') \,. \tag{4.6}$$

For the Navier-Stokes equation, we consider $u_{\varepsilon,0} \in H$ as an initial condition and $g = N_\varepsilon \nabla\phi \in L^\infty(Q_T) \subset L^2(0,T,V')$ as a second member. By taking $\psi = u_\varepsilon$ as a test function in (3.3), by using the fact that

$$(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon) = \frac{1}{2}\frac{d}{dt}||u_\varepsilon(t)||_H^2 \text{ and } b(u_\varepsilon,u_\varepsilon,u_\varepsilon) = < (u_\varepsilon \cdot \nabla)u_\varepsilon, u_\varepsilon >_{V',V} = 0$$

and by integrating between 0 and t, one obtains

$$\frac{1}{2}\big|\big|u_\varepsilon(t)\big|\big|_H^2 + \alpha\int_0^t \big|\big|u_\varepsilon(s)\big|\big|_V^2 \, ds \le \frac{1}{2}\big|\big|u_\varepsilon(0)\big|\big|_H^2 + \int_0^t \big|\big|g(s)\big|\big|_{V'}\big|\big|u_\varepsilon(s)\big|\big|_V \, ds \,.$$

Consequently, by a simple application of the Young inequality, one can easily deduce that the sequence of solutions $(u_\varepsilon)_\varepsilon$ of (3.3)-(3.4) is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Therefore, there exists $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that as ε goes to 0,

$$u_\varepsilon \rightharpoonup u \text{ weakly-* in } L^\infty(0,T;H) \text{ and} \tag{4.7}$$

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;V) \,. \tag{4.8}$$

Remark 4.1. *In fact, there exist two limits u_1 and u_2 for these last convergences but one can easily prove that $u_1 = u_2 = u$. Indeed, by writing clearly the definition of each of weak and weakly-* convergence, one has,*

$$\int_0^T (u_1(t) - u_2(t), h(t))\, dt = 0, \; \forall h \in L^2(0,T;H) \subset L^1(0,T;H) \cap L^2(0,T;V')\,.$$

Next, it follows from the choice of $h = u_1 - u_2$ that $||u_1 - u_2||_{L^2(0,T;H)} = \int_0^T \big|\big|u_1(t) - u_2(t)\big|\big|_H^2 = 0$.

Moreover, due to the compacity Theorem of Aubin-Simon, the space $E_{2,1} = \{u_\varepsilon \in L^2(0,T;V);\, \frac{du_\varepsilon}{dt} \in L^1(0,T;V')\}$ is compactly injected in $L^2(0,T;H)$ (see [2], Theorem 2.5.15). Therefore, modulo a subsequence,

$$u_\varepsilon \longrightarrow u \text{ in } L^2(0,T;H), \text{ as } \varepsilon \to 0\,. \tag{4.9}$$

One can also deduce this last assertion by a compactness theorem involving fractional derivatives and the inverse Fourier transform of a function (see [23]).

Then, by choosing $\psi_1 = A_\varepsilon(N_\varepsilon) = A(N_\varepsilon) + \varepsilon N_\varepsilon$ as a test function in (3.1), one has

$$\int_0^T < \partial_t(N_\varepsilon), A_\varepsilon(N_\varepsilon) > dx + \iint_{Q_T} S(x)\nabla A_\varepsilon(N_\varepsilon)\cdot\nabla A_\varepsilon(N_\varepsilon)\, dxdt = \iint_{Q_T} N_\varepsilon u_\varepsilon \cdot \nabla A_\varepsilon(N_\varepsilon)\, dxdt$$
$$+ \iint_{Q_T} S(x)\chi(N_\varepsilon)\nabla C_\varepsilon \cdot \nabla A_\varepsilon(N_\varepsilon)\, dxdt + \iint_{Q_T} f(N_\varepsilon)A_\varepsilon(N_\varepsilon)\, dxdt\,.$$

Considering $\mathcal{A}(s) = \int_0^s A(r)\, dr$ and using (2.7), Young's inequality for $\nabla C_\varepsilon \cdot \nabla A_\varepsilon(N_\varepsilon)$, $u_\varepsilon \cdot \nabla A_\varepsilon(N_\varepsilon)$ and the uniform bound of χ, f and A_ε, imply that

$$\sup_{0\le t\le T}\int_\Omega \mathcal{A}(N_\varepsilon)(x,t)dx + \varepsilon \sup_{0\le t\le T}\int_\Omega \frac{|N_\varepsilon(x,t)|^2}{2}dx + \frac{1}{2}\iint_{Q_T} |\nabla A_\varepsilon(N_\varepsilon))|^2\, dxdt$$
$$+\frac{\varepsilon}{2}\iint_{Q_T} |\nabla N_\varepsilon|^2\, dxdt \le C\,, \tag{4.10}$$

where C is a constant independent of ε. Then we will deduce, as ε goes to 0 that

$$N_\varepsilon \rightharpoonup N \text{ weakly-* in } L^\infty(Q_T), \tag{4.11}$$

$$\sqrt{\varepsilon}N_\varepsilon \rightharpoonup 0 \text{ in } L^2(0,T;H^1(\Omega)),\; A(N_\varepsilon) \rightharpoonup \Gamma_1 \text{ in } L^2(0,T;H^1(\Omega)). \tag{4.12}$$

Using the weak formulation (3.1), (4.2) and (4.10),

$$\Big|\int_0^T < \partial_t N_\varepsilon, \psi_1 > \; dt\Big| \le ||\nabla A(N_\varepsilon)||_{L^2(Q_T)}||\nabla\psi_1||_{L^2(Q_T)} + ||\sqrt{\varepsilon}\nabla N_\varepsilon||_{L^2(Q_T)}||\nabla\psi_1||_{L^2(Q_T)}$$
$$+||S(x)\chi(N_\varepsilon)||_{L^\infty(Q_T)}||\nabla C_\varepsilon||_{L^2(Q_T)}||\nabla\psi_1||_{L^2(Q_T)} + ||u_\varepsilon||_{L^2(Q_T)}||\nabla\psi_1||_{L^2(Q_T)}$$
$$+||f(N_\varepsilon)||_{L^2(Q_T)}||\psi_1||_{L^2(Q_T)} \le C||\psi_1||_{L^2(0,T;H^1(\Omega))}\,,$$

where C is a constant independent of ε. Hence,

$$||\partial_t N_\varepsilon||_{L^2(0,T;(H^1(\Omega))')} \le C\,, \tag{4.13}$$

$$\frac{\partial N_\varepsilon}{\partial t} \rightharpoonup \frac{\partial N}{\partial t} \text{ in } L^2(0,T;(H^1(\Omega))')\,. \tag{4.14}$$

From (4.11), (4.14) and the compact injection $L^\infty(\Omega) \hookrightarrow (H^1(\Omega))'$, one can prove similarly to the assertion (3.39) that

$$N_\varepsilon \longrightarrow N \text{ in } C(0,T;(H^1(\Omega))') \text{ as } \varepsilon \to 0.$$

Moreover, $N \in C_w(0,T;L^2(\Omega))$ (see [2], Lemma 2.5.7). With the same guidelines of the proof of the estimate (3.37) in the Proposition 3.6, one can also prove that $N(0,x) = N_0(x)$.

In addition to that, it is easy to prove that $A(N_\varepsilon)$ is bounded uniformly in $\mathcal{W} = \{N \in L^2(0,T;H^1(\Omega)),\, \frac{\partial N}{\partial t} \in L^2(0,T;(H^1(\Omega))')\}$ and as we have $\mathcal{W} \hookrightarrow L^2(Q_T)$ is compact (see [2], Theorem 2.5.12) then we deduce that

$$\text{there exists a subsequence of } (N_\varepsilon)/A(N_\varepsilon) \longrightarrow \Gamma_1 \text{ in } L^2(Q_T)\,. \tag{4.15}$$

But as A is strictly monotone, there exists N such that

$$\Gamma_1 = A(N)\,. \tag{4.16}$$

Thus,

$$A(N_\varepsilon) \longrightarrow A(N) \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T\,. \tag{4.17}$$

Furthermore, as A^{-1} is well defined and continuous, we apply the dominated convergence theorem to $N_\varepsilon = A^{-1}(A(N_\varepsilon))$ to obtain,

$$N_\varepsilon = A^{-1}(A(N_\varepsilon)) \longrightarrow N \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T\,. \tag{4.18}$$

Lemma 4.2. *The sequence $(C_\varepsilon)_\varepsilon$ converges strongly to C in $L^2(0,T;H^1(\Omega))$ as $\varepsilon \longrightarrow 0$.*

Proof. Indeed, subtracting the relations satisfied by $(N_\varepsilon, C_\varepsilon)$ and (N, C), we have

$$\partial_t(C_\varepsilon - C) + \nabla\cdot M(x)\nabla(C_\varepsilon - C) + [u_\varepsilon\cdot\nabla(C_\varepsilon - C) + (u_\varepsilon - u)\cdot\nabla C] = (N_\varepsilon - N)k(C) + N_\varepsilon(k(C_\varepsilon) - k(C))\,.$$

Taking $\psi = C_\varepsilon - C$ as a test function and using (2.7), (3.13), (3.14) (as $\nabla\cdot(u_\varepsilon - u) = 0$) and Young's inequality, one can conclude the existence of two positive constants c_1 and c_2 such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |C_\varepsilon - C|^2\,dx + c_M\int_\Omega |\nabla(C_\varepsilon - C)|^2\,dx &\le c_1\int_\Omega |N_\varepsilon - N|^2\,dx\\ &+c_2\int_\Omega |C_\varepsilon - C|^2\,dx + M\int_\Omega |u_\varepsilon - u||\nabla C_\varepsilon|\,dx\,.\end{aligned}$$

Integrating this inequality on $[0,T]$ and using the inequality of Cauchy-Schwarz, one has

$$\begin{aligned}c_M\iint_{Q_T}|\nabla(C_\varepsilon - C)|^2\,dxdt &\le c_1\iint_{Q_T}|N_\varepsilon - N|^2\,dxdt + c_2\iint_{Q_T}|C_\varepsilon - C|^2\,dxdt\\ &+M\big(\iint_{Q_T}|u_\varepsilon - u|^2\,dxdt\big)^{\frac12}\big(\iint_{Q_T}|\nabla C_\varepsilon|^2\,dxdt\big)^{\frac12}\,.\end{aligned}$$

Due to (4.1) and the strong convergence of N_ε, C_ε and u_ϵ respectively to N, C and u in $L^2(Q_T)$, one deduces the strong convergence of $(\nabla C_\varepsilon)_\varepsilon$ to ∇C in $L^2(Q_T)$. $\square$

4.2. **Passing to the limit.** Let us now tend the regularization parameter ε to 0. Hence,

$$\int_0^T <\partial_t(N_\varepsilon),\psi_1> dt \longrightarrow \int_0^T <\partial_t N,\psi_1> dt \text{ using (4.14).}$$

$$\iint_{Q_T} S(x)\nabla A_\varepsilon(N_\varepsilon)\cdot\nabla\psi_1\,dxdt \longrightarrow \iint_{Q_T} S(x)\nabla A(N)\cdot\nabla\psi_1\,dxdt\,,$$

one defines $\nabla : L^2(0,T;H^1(\Omega)) \longrightarrow L^2(Q_T)$ as a linear continuous application and therefore weakly continuous. Consequently, using (4.12) and (4.16), we have: $\nabla A_\varepsilon(N_\varepsilon) \rightharpoonup \nabla A(N)$ in $L^2(Q_T)$. Again, the weakly continuous application: $v \to S(x)v$ implies that $S(x)\nabla A_\varepsilon(N_\varepsilon) \rightharpoonup S(x)\nabla A(N)$ in $L^2(Q_T)$.

$$\iint_{Q_T} S(x)\chi(N_\varepsilon)\nabla C_\varepsilon\cdot\nabla\psi\,dxdt \longrightarrow \iint_{Q_T} S(x)\chi(N)\nabla C\cdot\nabla\psi\,dxdt\,,$$

this previous convergence follows from the L^∞-bound of $\chi(N_\varepsilon)$ and Lemma 4.2. Next,

$$\iint_{Q_T} N_\varepsilon u_\varepsilon\cdot\nabla\psi\,dxdt \longrightarrow \iint_{Q_T} Nu\cdot\nabla\psi\,dxdt\,,$$

indeed,

$$I = \iint_{Q_T}(N_\varepsilon u_\varepsilon\cdot\nabla\psi - Nu\cdot\nabla\psi)\,dxdt = \iint_{Q_T}(N_\varepsilon - N)u_\varepsilon\cdot\nabla\psi\,dxdt + \iint_{Q_T} N(u_\varepsilon - u)\cdot\nabla\psi\,dxdt\,.$$

It follows from the dominated convergence theorem of Lebesgue and (4.9) that

$$I \le ||(N_\varepsilon - N)\nabla\psi||_{L^2(Q_T)}||u_\varepsilon||_{L^2(Q_T)} + ||u_\varepsilon - u||_{L^2(Q_T)}||N\nabla\psi||_{L^2(Q_T)} \longrightarrow 0\,.$$

Finally, the Lipschitz continuity of f and the strong convergence of N_ε to N in $L^2(Q_T)$ yield

$$\iint_{Q_T}\Big(f(N_\varepsilon) - f(N)\Big)\psi\,dxdt \le L||N_\varepsilon - N||_{L^2(Q_T)}||\psi||_{L^2(Q_T)} \longrightarrow 0\,.$$

4.3. **Conclusion.** We have thus identified N, C and u as the components of a weak solution of the degenerate system (1.1) in the sense of Definition 2.1.

Moreover, one can write (2.13) as

$$\frac{d}{dt} < u, \psi >=< g - \Delta u - B(u), \psi >_{V',V} , \forall \psi \in V. \tag{4.19}$$

Since the Laplacien operator $-\Delta$ is linear and continuous from V into V' and $u \in L^2(0,T;V)$, then the function $-\Delta u$ belongs to $L^2(0,T;V')$. Next, since the form $b(u,u,w) =< (u\cdot\nabla)u, w >_{V',V}=< B(u), w >_{V',V}$ is trilinear continuous on V (in space dimension $d \leq 4$), so that $||B(u)||_{V'} \leq c||u||_V^2$. Consequently, the function $B(u)$ belongs to $L^1(0,T;V')$. As a conclusion,

$$\frac{du}{dt} \text{ belongs to } L^1(0,T;V')\,.$$

Hence the end of the proof of Theorem 2.3.

Remark 4.3. *Through the same guidelines, one can prove the existence of a weak solution to the system* (1.4) *in the sense of Definition 2.2. The unique difference is that coupling with linear Stokes equation* ($B(u) = 0$ *in* (4.19)) *implies that* $\frac{du}{dt}$ *belongs to* $L^2(0,T;V')$ *and consequently* u *belongs to* $C(0,T;H)$ (*see* [23]).
Furthermore, according to $\big[$[14], *Theorem* 6, *page* 100$\big]$, *a different* L^p*-regularity assumed on the initial data* u_0 *gives better regularity on* u *i.e.* $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ *and* $\nabla\cdot u_0 = 0$ *implies that* $u \in W_p^{2,1}(Q_T)$ *for* $1 < p < \infty$ *and*

$$\int_0^T \big(||u||^p_{W^{2,p}(\Omega)} + ||\partial_t u||^p_{L^2(\Omega)}\big)\, dt \leq C\big(||u(0)||^p_{W^{1,p}(\Omega)} + \int_0^T ||N\nabla\phi||^p_{L^p(\Omega)}\, dt\big)\,, \tag{4.20}$$

where C *is a positive constant. Consequently, by the Sobolev embedding, there is* $p \geq 2$ *large enough such that* ∇u *belongs to* $L^2(0,T;L^\infty(\Omega))$.

Remark 4.4. *This remark is devoted to the chemical equation of the system* (1.4). *If* $C_0 \in W^{2-\frac{2}{p},p}(\Omega)$, *the second member is uniformly bounded in* Q_T *and the regularity of* u *is given in* (4.20), *then classical parabolic regularity results* $\big($*see* [15], *chapter* 4$\big)$ *imply that* C *belongs to* $L^p(0,T;W^{2,p}(\Omega))$ *for each* $1 < p < \infty$ *and consequently there is* $p \geq 2$ *large enough such that* ∇C *belongs to* $L^2(0,T;L^\infty(\Omega))$.

4.4. **The pressure.** For the Navier-Stokes equation, we introduce $\tilde{u}(t)= \int_0^t u(s)\,ds$, $G(t)= \int_0^t g(s)\,ds$ and $\beta(t)= \int_0^t (u(s)\cdot\nabla)u(s)\,ds$ with $\tilde{u}, G$ and $\beta \in C(0,T;(H^1(\Omega))')$. Till now, we have found a weak solution u in the sense of Formulation (3.3)-(3.4) such that (4.19) hold true. It follows from integrating (4.19) over $[0,T]$ that $< u(t) - u_0 - \Delta\tilde{u} + \beta - G,\ \psi >= 0$, $\forall\psi \in V$, $\forall t \in [0,T]$. Rham Theorem (see [23], chapter 1) implies that there exists $P(t) \in L^2_0(\Omega)$, $\forall t \in [0,T]$ such that $u(t) - u_0 - \Delta\tilde{u} + \beta + \nabla P(t) = G(t)$, where

$$L^2_0(\Omega) = \{w \in L^2(\Omega),\ \int_\Omega w\,dx = 0\}\,. \tag{4.21}$$

Therefore $\nabla P \in C(0,T;(H^1(\Omega))')$ and then $P \in C(0,T;L^2_0(\Omega))$. Deriving with respect to t in the sense of distributions to obtain: $\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = g$ where $p = \frac{dP}{dt} \in W^{-1,\infty}(0,T;L^2_0(\Omega))$. The same work is done for Stokes equation in the system (1.4) (it suffices to consider $\beta = 0$).

5. Proof of Theorem 2.4

Under additional assumptions (2.10) and (2.14), the proof of the uniqueness statement relies on a duality technique. One should mention that Remarks 4.3 and 4.4 are verified due to the regularities imposed on the initial data in (2.10). We consider the subset $L^2_0(\Omega)$ of $L^2(\Omega)$ defined in (4.21) and we denote by $\mathcal{N}w \in H^2(\Omega)\cap L^2_0(\Omega)$ the unique solution to

$$\left\{\begin{array}{rcl} -\nabla\cdot\big(S(x)\nabla\mathcal{N}w\big) &=& w \\ S(x)\nabla\mathcal{N}w\cdot\eta &=& 0 \end{array}\right. . \tag{5.1}$$

Let (N_1, C_1, u_1) and (N_2, C_2, u_2) be two weak solutions of the system (1.4) in the sense of Definition 2.2. We fix $T > 0$ and we consider for $(t,x) \in [0,T]\times\Omega$,

$$N(t,x) = N_1(t,x) - N_2(t,x),\ C(t,x) = C_1(t,x) - C_2(t,x),\ U(t,x) = u_1(t,x) - u_2(t,x)\,.$$

We will start by subtracting equations related to the solutions u_1 and u_2 in the system (1.4) and by choosing $\psi = U$ as a test function to obtain

$$< \partial_t U, U > + \int_\Omega \nabla U^2 dx = - \int_\Omega N \nabla\phi \cdot U \, dx .$$

The variational problem associated to the dual problem (5.1) and Leibniz formula imply that

$$\frac{d}{dt} ||U(t)||^2_{L^2(\Omega)} + ||\nabla U(t)||^2_{L^2(\Omega)} = - \int_\Omega S(x) \nabla \mathcal{N} N \cdot \nabla(\nabla\phi \cdot U) \, dx$$

$$= - \int_\Omega S(x) \nabla \mathcal{N} N \cdot \Big[(\nabla\phi \cdot \nabla) U + (U \cdot \nabla)\nabla\phi + \nabla\phi \times \mathrm{curl}(U) + U \times \mathrm{curl}(\nabla\phi) \Big] \, dx .$$

where $a \times b$ denotes the vector product of two vectors a and b. Due to the following properties: $\mathrm{curl}(\nabla\phi) = 0$, $||a \times b||_{L^2(\Omega)} \leq ||a||_{L^2(\Omega)} ||b||_{L^2(\Omega)}$, $||\mathrm{curl}(U)||_{L^2(\Omega)} \lesssim ||\nabla U||_{L^2(\Omega)}$ (where a $\lesssim$ b means that there exists a positive constant c' such that $a \leq c' b$), Poincaré and Young's inequalities, one has

$$\begin{aligned} \frac{d}{dt} ||U(t)||^2_{L^2(\Omega)} + ||\nabla U(t)||^2_{L^2(\Omega)} &\leq ||S(x)||_{L^\infty(\Omega)} \Big((1 + c_P + c') ||\nabla\phi||_{W^{1,\infty}(\Omega)} \Big) ||\nabla \mathcal{N} N(t)||_{L^2(\Omega)} ||\nabla U(t)||_{L^2(\Omega)} \\ &\leq \Big(\frac{1}{\delta} c_1^2 ||S(x)||^2_{L^\infty(\Omega)} ||\nabla\phi||^2_{W^{1,\infty}(\Omega)} \Big) ||\nabla \mathcal{N} N(t)||^2_{L^2(\Omega)} + \delta ||\nabla U(t)||^2_{L^2(\Omega)} , \end{aligned} \tag{5.2}$$

where c_P, c' and c_1 are positive constants.

Next, we subtract equations related to the solutions C_1 and C_2 of the chemo-attractant equation and we choose $\psi_2 = C$ as a test function, then

$$\begin{aligned} < \partial_t C, C > + \int_\Omega M(x) \nabla C^2 \, dx + \int_\Omega (U \cdot \nabla C_1) C \, dx + \int_\Omega (u_2 \cdot \nabla C) C \, dx = - \int_\Omega (k(C_1) - k(C_2)) N_1 C \, dx \\ - \int_\Omega k(C_2) N C \, dx = - \int_\Omega (k(C_1) - k(C_2)) N_1 C \, dx - \int_\Omega S(x) \nabla \mathcal{N} N \cdot \nabla (k(C_2) C) \, dx . \end{aligned}$$

It follows from (2.7), (3.14), (2.3) and Young's inequality that

$$\begin{aligned} \frac{d}{dt} ||C(t)||^2_{L^2(\Omega)} + c_M ||\nabla C(t)||^2_{L^2(\Omega)} &\leq ||C_1||_{L^\infty(\Omega)} ||U(t)||_{L^2(\Omega)} ||\nabla C(t)||_{L^2(\Omega)} + c_k \Big(||C(t)||^2_{L^2(\Omega)} \\ ||S||_{L^\infty(\Omega)} ||\nabla \mathcal{N} N(t)||_{L^2(\Omega)} ||\nabla C(t)||_{L^2(\Omega)} &+ ||\nabla C_2||_{L^\infty(\Omega)} ||S||_{L^\infty(\Omega)} ||\nabla \mathcal{N} N(t)||_{L^2(\Omega)} ||C(t)||_{L^2(\Omega)} \Big) \\ &\leq M^2 \delta ||\nabla C(t)||^2_{L^2(\Omega)} + \frac{1}{\delta} ||U(t)||^2_{L^2(\Omega)} + c_k ||C(t)||^2_{L^2(\Omega)} + \frac{1}{\delta} ||S||^2_{L^\infty(\Omega)} ||\nabla \mathcal{N} N(t)||^2_{L^2(\Omega)} \\ &+ \delta c_k^2 |||\nabla C(t)||^2_{L^2(\Omega)} + \delta ||S||^2_{L^\infty(\Omega)} c_k^2 ||\nabla C_2||^2_{L^\infty(\Omega)} ||\nabla \mathcal{N} N(t)||^2_{L^2(\Omega)} + \frac{1}{\delta} ||C(t)||^2_{L^2(\Omega)} , \end{aligned} \tag{5.3}$$

where c_k is a positive constant.

Since we have $-\nabla \cdot \Big(S(x) \nabla \partial_t (\mathcal{N} N) \Big) = \partial_t N$ in $(H^1(\Omega))'$ and due to the symmetry of S, one can write

$$\int_\Omega S(x) \nabla \mathcal{N} N(t) \cdot \nabla \mathcal{N} N(t) \, dx = \int_\Omega S(x) \nabla \mathcal{N} N(0) \cdot \nabla \mathcal{N} N(0) \, dx + 2 \int_0^t < \frac{\partial N}{\partial t}, \mathcal{N} N > \, ds.$$

The subtraction of the two variational equalities related to the weak solutions N_1 and N_2 in the sense of Definition 2.2 leads to

$$\begin{aligned} \int_0^t < \frac{\partial N}{\partial t}, \mathcal{N} N > \, ds &= - \int_0^t \int_\Omega S(x) \nabla (A(N_1) - A(N_2)) \cdot \nabla \mathcal{N} N \, dx ds \\ &+ \int_0^t \int_\Omega S(x) \big(\chi(N_1) - \chi(N_2) \big) \nabla C_1 \cdot \nabla \mathcal{N} N \, dx ds + \int_0^t \int_\Omega S(x) \chi(N_2) \nabla C \cdot \nabla \mathcal{N} N \, dx ds \\ &+ \int_0^t \int_\Omega N_1 U \cdot \nabla \mathcal{N} N \, dx ds + \int_0^t \int_\Omega N u_2 \cdot \nabla \mathcal{N} N \, dx ds + \int_0^t \int_\Omega \big(f(N_1) - f(N_2) \big) \mathcal{N} N \, dx ds . \end{aligned} \tag{5.4}$$

As we have $|N_1| < 1$, then

$$\int_\Omega N_1 U \cdot \nabla \mathcal{N} N \, dx \leq ||\nabla \mathcal{N} N(t)||_{L^2(\Omega)} ||U(t)||_{L^2(\Omega)} . \tag{5.5}$$

It next follows from the dual problem (5.1) and again from Leibniz formula that

$$\int_\Omega N(u_2\cdot\nabla\mathcal{N}N)\,dx=\int_\Omega S(x)\nabla\mathcal{N}N\cdot\nabla(u_2\cdot\nabla\mathcal{N}N)\,dx$$

$$=\int_\Omega S(x)\nabla\mathcal{N}N\cdot\Big[(u_2\cdot\nabla)\nabla\mathcal{N}N+(\nabla\mathcal{N}N\cdot\nabla)u_2+\nabla\mathcal{N}N\times\mathrm{curl}(u_2)+u_2\times\mathrm{curl}(\nabla\mathcal{N}N)\Big]\,dx\,. \tag{5.6}$$

The property $\mathrm{curl}(\nabla\mathcal{N}N)=0$ allows us to omit the last integral. Then, we write the first integral as

$$\int_\Omega S(x)\nabla\mathcal{N}N\cdot(u_2\cdot\nabla)\nabla\mathcal{N}N\,dx=\sum_i\sum_{j,k}\int_\Omega u_iS_{j,k}\frac{\partial}{\partial x_k}(\mathcal{N}N)\cdot\frac{\partial}{\partial x_i}\big(\frac{\partial}{\partial x_j}(\mathcal{N}N)\big)\,dx\,,$$

where $u_i\in L^\infty(\Omega)$ and the coefficients $S_{j,k}$ are assumed to be of class C^1 in (2.10). Due to the Green formula, $\nabla\cdot u_2=0$ and the symmetry of the tensor S, one obtains

$$2\int_\Omega S(x)\nabla\mathcal{N}N\cdot(u_2\cdot\nabla)\nabla\mathcal{N}N\,dx=-\int_\Omega(\nabla\cdot u_2)S(x)\nabla\mathcal{N}N\cdot\nabla\mathcal{N}N\,dx-\int_\Omega\big(u_2\cdot\nabla(S(x))\big)\nabla\mathcal{N}N\cdot\nabla\mathcal{N}N\,dx \tag{5.7}$$
$$\le||u_2||_{L^\infty(\Omega)}||\nabla S||_{L^\infty(\Omega)}||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,.$$

Next, one deduces from the affine function f and the dual problem (5.1) that

$$\int_\Omega\big(f(N_1)-f(N_2)\big)\mathcal{N}N\,dx\le\alpha||S||_{L^\infty(\Omega)}||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,. \tag{5.8}$$

Recalling (5.5), (5.6), (5.7) and (5.8), we deduce from the equation (5.4), Cauchy Schwarz and Young's inequality that

$$\int_\Omega S(x)\nabla\mathcal{N}N(t)\cdot\nabla\mathcal{N}N(t)\,dx\le-2\int_0^t\int_\Omega(N_1-N_2)\big(A(N_1)-A(N_2)\big)\,dxds+2\delta\int_0^t\int_\Omega(\chi(N_1)-\chi(N_2))^2\,dxds \tag{5.9}$$
$$+\frac{2}{\delta}\int_0^t||S||^2_{L^\infty(\Omega)}||\nabla C_1||^2_{L^\infty(\Omega)}||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}ds+2\delta c_\chi^2\int_0^t||\nabla C(t)||^2_{L^2(\Omega)}\,ds+\frac{2}{\delta}\int_0^t||S||^2_{L^\infty(\Omega)}||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,ds$$
$$+2\delta\int_0^t||U(t)||^2_{L^2(\Omega)}+\frac{2}{\delta}\int_0^t||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}ds+\int_0^t||u_2||_{L^\infty(\Omega)}||\nabla S||_{L^\infty(\Omega)}||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,ds$$
$$+2\int_0^t||S||^2_{L^\infty(\Omega)}\big(||\nabla u_2(t)||^2_{L^\infty(\Omega)}+||\nabla u_2(t)||^2_{L^2(\Omega)}\big)||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,ds+2\alpha\int_0^t||S||_{L^\infty(\Omega)}||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,ds\,.$$

Now, we integrate the previous inequalities (5.2) and (5.3) with respect to time and we sum the integrated inequalities with (5.9). Then, we consider $0<\delta<\min(\frac{1}{C_0},\,\frac{c_M}{2c_\chi^2+c_k^2+M^2},\,1)$ and we use (2.14) to deduce that

$$c_S||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}+||C(t)||^2_{L^2(\Omega)}+||U(t)||^2_{L^2(\Omega)}\le\int_0^t\big(\frac{1}{\delta}+c_k\big)||C(t)||^2_{L^2(\Omega)}\,ds$$
$$+\int_0^t\big(\frac{1}{\delta}+2\big)||U(t)||^2_{L^2(\Omega)}\,ds+\int_0^t\frac{1}{\delta}\Big[2+2\alpha||S||_{L^\infty(\Omega)}+||u_2(t)||_{L^\infty(\Omega)}||\nabla S||_{L^\infty(\Omega)}$$
$$+||S||^2_{L^\infty(\Omega)}\Big(3+2||\nabla C_1(t)||^2_{L^\infty(\Omega)}+2\big(||\nabla u_2(t)||^2_{L^\infty(\Omega)}+||\nabla u_2(t)||^2_{L^2(\Omega)}\big)+c_k^2||\nabla C_2(t)||^2_{L^\infty(\Omega)}$$
$$+c_1^2||\nabla\phi||^2_{W^{1,\infty}(\Omega)}\Big)\Big]||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}\,ds\,.$$

According to remarks 4.3 and 4.4, one deduces that

$$||\nabla\mathcal{N}N(t)||^2_{L^2(\Omega)}+||C(t)||^2_{L^2(\Omega)}+||U(t)||^2_{L^2(\Omega)}\le\int_0^t\mu(s)\Big[||\nabla\mathcal{N}N||^2_{L^2(\Omega)}+||C||^2_{L^2(\Omega)}+||U||^2_{L^2(\Omega)}\Big]\,ds\,,$$

where $\mu(s)$ is a positive integrable function. The Gronwall Lemma (see [20]) then entails that $U(t)=C(t)=\nabla\mathcal{N}N(t)=0$ for every $t\in[0,T]$ and therefore the proof of Theorem 2.4 is achieved.

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