# Combined finite volume–finite element scheme for compressible two phase flow in porous media

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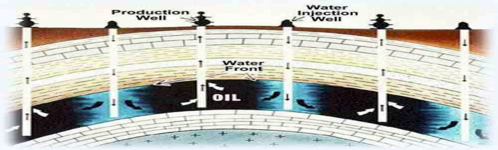
# Two compressible and immiscible flow

Secondary recovery of oil.

The secondary recovery consists to inject water (waterflooding) or gas into an oil reservoir through specially distributed injection wells. The purpose of this method is to displace hydrocarbons toward the production wells. The successive use of primary recovery and secondary recovery in an oil reservoir produces about 15 percent to 40 percent of the original oil in place.




Water–Oil or Water–Gas flow in porous media.

# Two compressible and immiscible flow

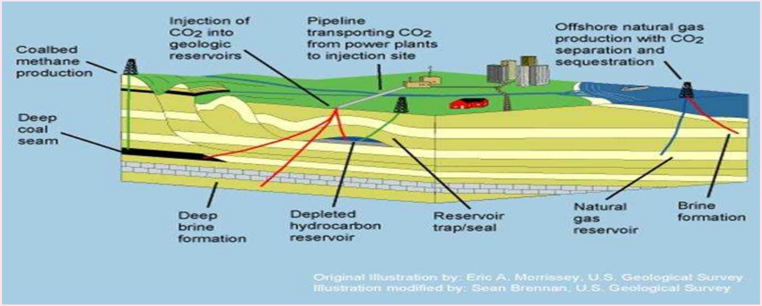
CO2 capture and storage.

The aim is to prevent the release of large quantities of CO2 into the atmosphere. The process consists of capturing waste CO2 and transporting it to a storage site.

Various forms have been conceived for storage CO2 into deep geological formations :

- CO2 is sometimes injected into declining oil fields to increase oil recovery.

This option is attractive because the geology of hydrocarbon reservoirs is generally well understood and storage costs may be partly offset by the sale of additional oil that is recovered.




Gas $\approx 90\%$ CO2 $\Longrightarrow$ Oil–Gas model.

## Two compressible immiscible flow

The formulation describing the immiscible displacement of two compressible fluids is given by the mass conservation of each phase. We consider a porous media saturated by a liquid and a gas fluids.

- Mass conservation of each phase : $\alpha = l, g$

$$\Phi \partial_t(\rho_\alpha(p_\alpha)s_\alpha) + \mathrm{div}(\rho_\alpha(p_\alpha)\mathbf{V}_\alpha) = f_\alpha$$

- Darcy's law for velocities

$$\mathbf{V}_\alpha = -\Lambda(x)M_\alpha(s_\alpha)\nabla p_\alpha$$

- $\Phi$ = porosity
  $\rho_\alpha(p_\alpha)$ = density of the $\alpha$ phase
  $s_\alpha$ = saturation of the $\alpha$ phase

  $p_\alpha$ = pressure of the $\alpha$ phase
  $\mathbf{V}_\alpha$ = velocity of the $\alpha$ phase
  $f_\alpha$ = source term

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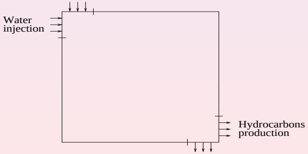
Boundary conditions : $\partial\Omega = \Gamma_l \cup \Gamma_{\text{imp}}$

$$\begin{cases} p_\alpha(t,x) = 0 & \text{on } (0,T)\times\Gamma_l, \\ \rho_\alpha \mathbf{V}_\alpha \cdot \mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{\text{imp}}. \end{cases}$$

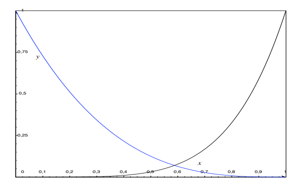


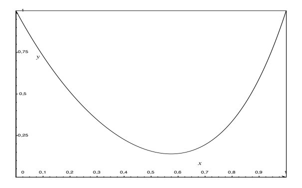
# Two compressible immiscible flow

Darcy law :

$$\mathbf{V}_\alpha(t,x) = -\mathbf{\Lambda}(x) M_\alpha(s_\alpha(t,x)) \nabla p_\alpha(t,x), \qquad \alpha = l, g$$

$\mathbf{\Lambda}(x)$ : Tensor of permeabilitiy, $p_\alpha$ : pressure of the $\alpha$ phase




$M_l(s_l)$ mobility of liquid phase
$M_g(s_l)$ mobility of gas phase

Totale mobility : $M = M_l + M_g \geq m_0$

The mobility of each phase vanishes in the region where the phase is missing $M_\alpha(s_\alpha = 0) = 0$.
It is not possible to control the gradient of pressures in the whole domain regardless of the presence or the disappearance of the phases (degenerate problem).

# General motivation

Is there a convergent scheme (FV, FE, dG,...) for the two compressible and immiscible flow ?
Develop and analysis of efficient numerical methods.

In M. Saad, B. Saad. Study of full implicit petroleum engineering finite volume scheme for compressible two phase flow in porous media, SIAM J. Numer. Anal., 51(1), pp. 716-741, 2013, we have shown for a homogeneous and isotropic medium that a upwind finite volume (FV4) is convergent under the assumption that the mesh satisfying the orthogonal property.

Here, we present a combined FV–FE method for inhomogeneous and anisotropic diffusion tensors and for general meshes

# A comparison between Finite volumes and Finite elements

Consider the diffusion-transport equation :

$$-\text{div}(\mathbf{\Lambda}\nabla u) + \text{div}(\mathbf{c}u) = 0 \text{ in } \Omega, u = 0 \text{ on } \Omega.$$

Finite elements. Find $u_h \in V_h \subset H_1^0(\Omega)$ such that

$$\int_\Omega \mathbf{\Lambda}\nabla u_h \cdot \nabla\phi_h - \int_\Omega \mathbf{c}u_h \cdot \nabla\phi_h = 0, \quad \forall\phi_h \in V_h$$

The + and - of FE

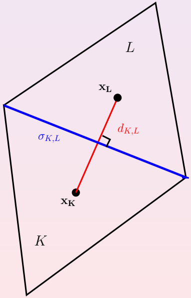
+ discretization of full diffusion tensor
+ no restrictions on the mesh
- instabilities in the transport dominated case (no maximum principle)

# FV versus FE for diffusion-transport equation

Finite volume. Let $\mathcal{T}_h$ be a partition of $\Omega$, for all $K \in \mathcal{T}_h$,

$$-\int_{\partial K} \mathbf{\Lambda} \nabla u \cdot \mathbf{n} + \int_{\partial K} u\mathbf{c} \cdot \mathbf{n} = 0,$$

The question is how to discretize the full diffusion tensor ? Consider the simple case $\mathbf{\Lambda} = \mathbf{Id}$ and admissible mesh of $\Omega$ in the sense of Eymard–Galllouët–Herbin.




$$\int_{\sigma_{K,L}} \nabla u \cdot \mathbf{n}_{K,L} \approx \frac{|\sigma_{K,L}|}{d_{K,L}}(u_L - u_K)$$

$$\int_{\sigma_{K,L}} u\,\mathbf{c}\cdot\mathbf{n}_{K,L} \approx u_K(\mathbf{c}\cdot\mathbf{n}_{K,L})^+ + u_L(\mathbf{c}\cdot\mathbf{n}_{K,L})^-$$

# FV versus FE for diffusion-transport equation

- The + and - of VF
  - restriction on the mesh
  - how to discretize the diffusion tensor
  + upwind techniques, no oscillations and stabilities in the transport dominated case

## FV versus FE for diffusion-transport equation

- The + and - of VF
  - restriction on the mesh
  - how to discretize the full tensor diffusion tensor
  
  + upwind techniques, no oscillations and stabilities in the transport dominated case
- Combined scheme : FE/ FV

$$\underbrace{-\mathrm{div}(\mathbf{\Lambda}\nabla u)}_{\text{Finite Element}} + \underbrace{\mathrm{div}(\mathbf{c}u)}_{\text{Finite Volume}} = 0$$

There is many schemes to handle with this problem. We present a combined FV–nonconforming FE scheme.

# Combined FV–nonconforming FE : primal mesh

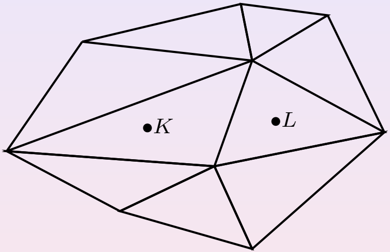



FIGURE : Primal mesh. Triangles $K, L \in T_h$

**Primal mesh.** we perform a triangulation $\mathcal{T}_h$ of the domain $\Omega$, consisting of closed simplices such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$.
We define

$$h := size(\mathcal{T}_h) = \max_{K\in\mathcal{T}_h} diam(K),$$

There exists a constant $\theta_{\mathcal{T}} > 0$

$$\max_{K\in\mathcal{T}_h} \frac{diam(K)}{\rho_K} \le \theta_{\mathcal{T}}, \forall h > 0\,, \quad (1)$$

where $\rho_K$ is the diameter of the largest ball inscribed in the simplex $K$.

# Combined FV–nonconforming FE : Dual mesh

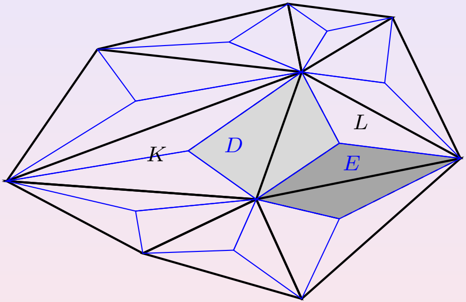



FIGURE : Dual mesh $D$,$D \in D_h$ , dual volumes associated with edges

**Dual mesh.** We define a dual partition $\mathcal{D}_h$ s.t. $\bar{\Omega} = \cup_{D \in \mathcal{D}_h} \bar{D}$. There is one dual element $D$ associated with each side $\sigma_D = \sigma_{K,L} \in \mathcal{E}_h$.

We construct it by connecting the barycenters of every $K \in \mathcal{T}_h$ that contains $\sigma_D$ through the vertices of $\sigma_D$.

# Combined FV–Nonconforming FE : Dual Mesh

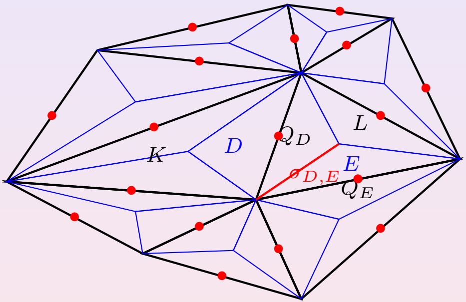



FIGURE : •The unknown are on edges

We use the following notations :

- $|D| = meas(D)$ and $|\sigma| = meas(\sigma)$.
- $Q_D$ the barycenter of the side $\sigma_D$.
- $\mathcal{N}(D)$ the set of neighbors of the volume $D$.
- $d_{D,E} := |Q_E - Q_D|$
- $\sigma_{D,E}$ : interface between $D$ and $E$
- $\eta_{D,E}$ : the unit normal vector to $\sigma_{D,E}$ outward to $D$.
- $\mathcal{D}_h^{int}$ and $\mathcal{D}_h^{ext}$ are respectively the set of all interior and boundary dual volumes.

## Combined FV–Nonconforming FE : Diffusion-Transport

We define the nonconforming finite-dimensional spaces :

$$X_h := \{\varphi_h \in L^2(\Omega); \varphi_h|_D \text{ is linear } \forall K \in \mathcal{T}_h, \varphi_h \text{ is continuous at } Q_D,\ D \in \mathcal{D}_h^{int}\},$$

$$X_h^0 := \{\varphi_h \in X_h; \varphi_h(Q_D) = 0,\ \forall D \in \mathcal{D}_h^{ext}\}.$$

$(\varphi_D)_{D\in\mathcal{D}_h}$ the basis of $X_h$ s.t. $\varphi_D(Q_E) = \delta_{DE},\ E \in \mathcal{D}_h$.
Combined scheme.

$$-\sum_{E\in\mathcal{N}(D)} \mathbf{\Lambda}_{D,E}(U_E - U_D) + \sum_{E\in\mathcal{N}(D)} G(U_D, U_E; \delta C_{D,E}) = 0$$

where the stiffness matrix is

$$\mathbf{\Lambda}_{D,E} = -\sum_{K\in\mathcal{T}_h} \int_K \mathbf{\Lambda}(x)\nabla\varphi_E \cdot \nabla\varphi_D \, dx \quad \text{(nonconforming FE)}$$

and the numerical flux $G$ is defined by

$$G(U_D, U_E; \delta C_{D,E}) = U_D(\delta C_{D,E})^+ + U_E(\delta C_{D,E})^- \quad \text{(upwind finite volume)}$$

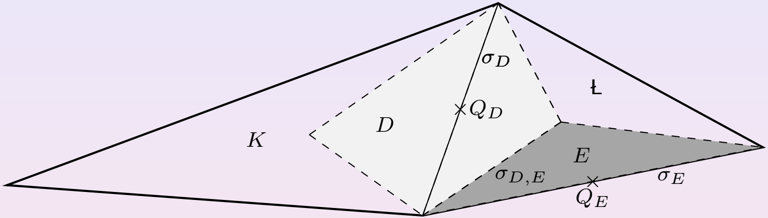
where $\delta C_{D,E} = \int_{\sigma_{D,E}} \mathbf{c} \cdot \mathbf{n}_{D,E} \, d\sigma$.

## Combined FV–nonconforming FE : Diffusion-transport

$$-\sum_{E\in\mathcal{N}(D)} \mathbf{\Lambda}_{D,E}(U_E - U_D) + \sum_{E\in\mathcal{N}(D)} G(U_D, U_E; \delta C_{D,E}) = 0$$

- P. Angot, V. Dolejsi, M. Feistauer and J. Felcman, Analysis of a combined barycentric finite volume-nonconforming finite element method for nonlinear convection-diffusion problems. Appl.Math.,43(4), p. 263-310, 1998.
- R. Eymard, D. Hilhorst and M. Vohralik, A combined finite volume-nonconforming/mixed hybrid finite element scheme for degenerate parabolic problems. Numer.Math., 105 : p. 73-131, 2006.

# Nonconforming FE/Implicit upwind scheme




We integrate the mass conversation law over the diamond $D$

$$\Phi\partial_t(\rho_\alpha(p_\alpha)s_\alpha) - \mathrm{div}(\Lambda(x)M_\alpha(s_\alpha)\nabla p_\alpha) = f_\alpha$$

and we use :

- Fully implicit scheme
- The mobility of each phase is decentred according to discrete gradient of the pressure on the interface $\sigma_{D,E}$
- Nonconforming FE for permeability tensor

For $\alpha = l, g$, for all $D \in \mathcal{D}_h$, for all $n \geq 1$

$$|D|\,\phi_D \frac{\rho_\alpha(p^n_{\alpha,D})s^n_{\alpha,D} - \rho_\alpha(p^{n-1}_{\alpha,D})s^{n-1}_{\alpha,D}}{\delta t} - \sum_{E \in \mathcal{N}(D)} \rho^n_{\alpha,D,E}\, M_\alpha(s^n_{\alpha,D,E})\, \Lambda_{D,E}\, \delta^n_{D,E}(p_\alpha) = f^n_{\alpha,D} \qquad (2)$$

This system is completed by the capillary pressure

$$p_c(s^n_{l,D}) = p^n_{g,D} - p^n_{l,D}. \qquad (3)$$

The approximation of each term is important to handle with the energy estimates.

$$|D|\,\phi_D \frac{\rho_\alpha(p^n_{\alpha,D})s^n_{\alpha,D} - \rho_\alpha(p^{n-1}_{\alpha,D})s^{n-1}_{\alpha,D}}{\delta t} - \sum_{E\in\mathcal{N}(D)} \rho^n_{\alpha,D,E}\, M_\alpha(s^n_{\alpha,D,E})\, \Lambda_{D,E}\, \delta^n_{D,E}(p_\alpha) = f^n_{\alpha,D} \quad (4)$$

- Discrete Gradient of pressure

$$\delta^n_{D,E}(p_\alpha) = p^n_{\alpha,E} - p^n_{\alpha,D}$$

$$|D|\,\phi_D \frac{\rho_\alpha(p^n_{\alpha,D})s^n_{\alpha,D} - \rho_\alpha(p^{n-1}_{\alpha,D})s^{n-1}_{\alpha,D}}{\delta t} - \sum_{E\in\mathcal{N}(D)} \rho^n_{\alpha,D,E}\, M_\alpha(s^n_{\alpha,D,E})\, \Lambda_{D,E}\, \delta^n_{D,E}(p_\alpha) = f^n_{\alpha,D} \quad (4)$$

- Permeability on interfaces by FE

$$\Lambda_{D,E} = -\sum_{K\in\mathcal{T}_h} \int_K \mathbf{\Lambda}(x)\nabla\varphi_E \cdot \nabla\varphi_D \, dx$$

$$|D|\,\phi_D \frac{\rho_\alpha(p^n_{\alpha,D})s^n_{\alpha,D} - \rho_\alpha(p^{n-1}_{\alpha,D})s^{n-1}_{\alpha,D}}{\delta t} - \sum_{E\in\mathcal{N}(D)} \rho^n_{\alpha,D,E}\, M_\alpha(s^n_{\alpha,D,E})\, \Lambda_{D,E}\, \delta^n_{D,E}(p_\alpha) = f^n_{\alpha,D} \qquad (4)$$

- Upwind technics for the mobilities

  The numerical fluxes, where $M_\alpha(s^n_{\alpha,D,E})$ denote the upwind discretization of $M_\alpha(s_\alpha)$ on the interface $\sigma_{D,E}$ as

$$M_\alpha(s^n_{\alpha,D,E}) = \begin{cases} M_\alpha(s^n_{\alpha,D}) \text{ if } \left(p^n_{\alpha,E} - p^n_{\alpha,D}\right) \le 0, \\ M_\alpha(s^n_{\alpha,E}) \text{ otherwise} \end{cases} \qquad (5)$$

$$|D|\,\phi_D \frac{\rho_\alpha(p^n_{\alpha,D}) s^n_{\alpha,D} - \rho_\alpha(p^{n-1}_{\alpha,D}) s^{n-1}_{\alpha,D}}{\delta t} - \sum_{E\in\mathcal{N}(D)} \rho^n_{\alpha,D,E}\, M_\alpha(s^n_{\alpha,D,E})\, \Lambda_{D,E}\, \delta^n_{D,E}(p_\alpha) = f^n_{\alpha,D} \quad (4)$$

**Mean value of densities on interfaces**

The mean value of the density of each phase on the interfaces is not classical since it is given as

$$\frac{1}{\rho^n_{\alpha,D,E}} = \begin{cases} \frac{1}{p^n_{\alpha,E}-p^n_{\alpha,D}} \displaystyle\int_{p^n_{\alpha,D}}^{p^n_{\alpha,E}} \frac{1}{\rho_\alpha(\zeta)}\, d\zeta & \text{if } p^n_{\alpha,D} \neq p^n_{\alpha,E}, \\ \frac{1}{\rho^n_{\alpha,D}} & \text{otherwise.} \end{cases} \quad (5)$$

- Source term

$$f^n_{\alpha,D} = -\,|D|\,\rho_l(p^n_{l,D}) s^n_{l,D} f^n_{P,D} + |D|\,\rho_l(p^n_{l,D}) (s^I_{l,D})^n f^n_{I,D},$$

## Convective term : $G^n_{\alpha,D,E} \approx -M_\alpha(s_\alpha)\nabla p_\alpha \cdot \mathbf{n}$ on $\sigma_{D,E}$

Upwind scheme according to $-\nabla p_\alpha \cdot \mathbf{n}$ on interfaces

$$\begin{aligned} G^n_{\alpha,D,E} &= -M_\alpha(s^n_{\alpha,D,E})\delta^n_{D,E}(p_\alpha) \\ &= -M_\alpha(s^n_{\alpha,E})\,(\delta^n_{D,E}(p_\alpha))^+ + M_\alpha(s^n_{\alpha,D})(\delta^n_{D,E}(p_\alpha))^- \\ &= G_\alpha(s^n_{\alpha,D}, s^n_{\alpha,E}, \delta^n_{D,E}(p_\alpha)) \end{aligned}$$

### The scheme reads to

$$|D|\,\phi_D \frac{\rho_\alpha(p^n_{\alpha,D})s^n_{\alpha,D} - \rho_\alpha(p^{n-1}_{\alpha,D})s^{n-1}_{\alpha,D}}{\delta t} - \sum_{E\in\mathcal{N}(D)} \rho^n_{\alpha,D,E}\,\Lambda_{D,E} G^n_{\alpha,D,E} = f^n_{\alpha,D}$$

Main properties on the numerical flux functions $G_\alpha$

- Monotony : $s \mapsto G_\alpha(s,\cdot,\cdot)$ is non-decreasing, $s \mapsto G_\alpha(\cdot,s,\cdot)$ is non-increasing
- Consistency : $G_\alpha(s,s,q) = -M_\alpha(s)\,q$
- Conservation : $G_\alpha(a,b,q) = -G_\alpha(b,a,-q)$
- Boundness : $|G_\alpha(a,b,q)| \le C\,\big(|a|+|b|\big)|q|$.

## Proposition (Maximum principle)

*Suppose* $\Lambda_{D,E} \geq 0$, *for all* $D, E$. *Let* $(s^0_{\alpha,D})_{D\in\mathcal{D}_h} \in [0,1]$. *Then, the saturation* $(s^n_{\alpha,D})_{D\in\mathcal{D}_h}$, *remains in* $[0,1]$ *for all* $D \in \mathcal{D}_h$, $n \in \{1,\ldots,N\}$.

Proof by induction on $n$. Suppose $s^{n-1}_{\alpha,D} \geq 0$ for all $D \in D_h$. Let $s^n_{l,D} = \min \{s^n_{l,E}\}_{E\in\mathcal{D}_h}$ and we seek that $s^n_{l,D} \geq 0$
Multiply the scheme by $-(s^n_{l,D})^-$, we obtain

$$- |D|\, \phi_D \frac{\rho_\alpha(p^n_{\alpha,D}) s^n_{\alpha,D} - \rho_l(p^{n-1}_{\alpha,D}) s^{n-1}_{\alpha,D}}{\delta t} (s^n_{\alpha,D})^-$$
$$- \sum_{E\in\mathcal{N}(D)} \rho^n_{\alpha,D,E}\, \Lambda_{D,E}\, \underbrace{G_\alpha(s^n_{\alpha,D}, s^n_{\alpha,E}; \delta^n_{D,E}(p_l))(s^n_{\alpha,D})^-}_{\leq\, 0,\text{ since } G_\alpha \text{ is monotone}} = \underbrace{f^n_{\alpha,D}(s^n_{l,D})^-}_{\leq 0}.$$

Then, we deduce that

$$\rho_\alpha(p^n_{\alpha,D}) |(s^n_{\alpha,D})^-|^2 + \rho_\alpha(p^{n-1}_{\alpha,D}) s^{n-1}_{\alpha,D} (s^n_{\alpha,D})^- \leq 0,$$

and $s^n_{\alpha,D} \geq 0$ for all $D \in D_h$.

# Energy estimates in the continuous case.

Let us recall how to obtain the energy estimates in the continuous case. For that, consider $g_\alpha(p_\alpha) = \displaystyle\int_0^{p_\alpha} \frac{1}{\rho_\alpha(z)} dz$ as a test function, then

$$\underbrace{\int_\Omega \phi(\partial_t(\rho_l s_l) g_l + \rho_g s_g) g_g)}_{=\frac{d}{dt}\int_\Omega H\,dx} + \underbrace{\int_\Omega \Lambda(x)(M_l(s_l)|\nabla p_l|^2 + M_g(s_g)|\nabla p_g|^2)}_{\color{red}{bounded}} \;\leq\; rhs$$

## Estimates on the velocities

$$\int_\Omega \Lambda(x)\Big(M_l(s_l)|\nabla p_l|^2 + M_g(s_g)|\nabla p_g|^2\Big) \leq C$$

we cannot control the gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing $M_\alpha(s_\alpha = 0) = 0$. So, we use the feature of global pressure to obtain uniform estimates on the gradient of the global pressure and on a function of the capillary term $\mathcal{B}$.

# Discrete Lemma

Continuous case.

The global pressure $p$ can be written as

$$p = p_l + \tilde{p}(s_l) = p_g + \bar{p}(s_l),$$

with the deviation pressures $\bar{p}$ and $\tilde{p}$ :

$$\tilde{p}(s_l) = -\int_0^{s_l} \frac{M_g(z)}{M(z)} p_c^{'}(z)\mathrm{d}z \text{ and } \overline{p}(s_l) = \int_0^{s_l} \frac{M_l(z)}{M(z)} p_c^{'}(z)\mathrm{d}z.$$

Total mobility : $M(s_l) = M_l(s_l) + M_g(s_l) \geq m_0 > 0$.

From the definition of the global pressure we have :

$$M_l(s_l)|\nabla p_l|^2 + M_g(s_l)|\nabla p_g|^2 = M(s_l)|\nabla p|^2 + \frac{M_l(s_l)M_g(s_l)}{M(s_l)}|\nabla p_c(s_l)|^2.$$

The control of velocities ensures the control of the gradient of the global pressure.

Discrete case.

In the discrete case, this relationship is not obtained in a straightforward way. This equality is replaced by three discrete inequalities which we state in the following lemma.

# Discrete lemma

Continuous case : $M(s_l)|\nabla p|^2 \leq M_l(s_l)|\nabla p_l|^2 + M_g(s_l)|\nabla p_g|^2$

## Lemma (Total mobility and global pressure)

$$M^n_{l,D|E} + M^n_{g,D|E} \geq m_0, \ \forall (D,E) \in \mathcal{E}, \ \forall n \in [0,N],$$

$$m_0 \Big(\delta^n_{D,E}(p)\Big)^2 \leq M^n_{l,D|E}\Big(\delta^n_{D,E}(p_l)\Big)^2 + M^n_{g,D|E}\Big(\delta^n_{D,E}(p_g)\Big)^2.$$

Continuous case : $|\nabla \mathcal{B}(s_l)|^2 = \frac{M_l M_g}{M}|\nabla p_c|^2 \leq M_l(s_l)|\nabla p_l|^2 + M_g(s_g)|\nabla p_g|^2.$

## Lemma (Capillary term)

$$(\delta^n_{D,E}(\mathcal{B}(s_l)))^2 \leq M^{n+1}_{g,K|L}\Big(\delta^n_{D,E}(p_g)\Big)^2 + M^{n+1}_{l,K|L}\Big(\delta^n_{D,E}(p_l)\Big)^2.$$

# A priori estimates

We show the discrete version of $\int_0^T \int_\Omega \Lambda(x) M_\alpha \nabla p_\alpha \cdot \nabla p_\alpha \, dtdx \leq C$ .

**Proposition (Discrete velocities)**

$$\sum_{n=0}^{N-1} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} M_\alpha(s^n_{\alpha,D|E}) \left| p^n_{\alpha,E} - p^n_{\alpha,D} \right|^2 \leq C. \tag{6}$$

The proof is based on the choice of the test function :

$$g_\alpha(p_{\alpha,K}) = \int_0^{p_{\alpha,K}} \frac{1}{\rho_\alpha(z)} dz$$

## Consequences

### Corollary (Discrete Gradients)

*From the preliminary lemmas, we have*

$$\sum_{n=0}^{N-1} \delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \delta^n_{D,E}(p)|^2 \leq C. \qquad \to p_h \in L^2(0,T,H^1(\Omega))$$

$$\sum_{n=0}^{N-1} \delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} (\delta^n_{D,E}(\mathcal{B}(s_l)))^2 \leq C. \qquad \to \mathcal{B}(s_{l,h}) \in L^2(0,T,H^1(\Omega))$$

# Compactness : translates in space and time estimates

Define the discrete function $U_{\alpha,h} = \rho_\alpha(p_{\alpha,h})s_{\alpha,h}$ constant per cylinder $(t^n, t^{n+1}) \times K$. We derive estimates on translates in space and time of the functions $\bar{U}_{\alpha,h}$ piecewise constant in $t$ and constant in $x$ for all $\mathcal{D}$.

## Lemma (Translates in space and in time)

$$\iint_{\Omega' \times (0,T)} \left|\bar{U}_{\alpha,h}(t, x+y) - \bar{U}_{\alpha,h}(t,x)\right| \mathrm{d}x\mathrm{d}t \leq \omega(|y|),$$

$$\iint_{\Omega \times (0,T-\tau)} \left|\bar{U}_{\alpha,h}(t+\tau, x) - \bar{U}_{\alpha,h}(t,x)\right|^2 dx\, dt \leq \tilde{\omega}(\tau),$$

*where* $y \in \mathbb{R}^3$, $\tau \in (0,T)$, $\Omega' = \{x \in \Omega,\ [x, x+y] \subset \Omega\}$ *and* $\omega$ *satisfies* $\lim_{|y|\to 0} \omega(|y|) = 0$ *and* $\lim_{\tau \to 0} \tilde{\omega}(\tau) = 0$.

## Strong convergence

The sequence $\bar{U}_{\alpha,h}$ is relatively compact in $L^1(Q_T)$, $\alpha = l, g$.

Using Kolmogorov compactness theorem.

## Convergence of FV–FE scheme

### Theorem

*The sequence* $(p_{l,h}, p_{g,h})$ *converges to* $(p_l, p_g)$ *(*$h \longrightarrow 0$*) :*

$$p_\alpha \in L^2(Q_T)), \qquad 0 \le s_\alpha \le 1 \text{ a.e in } Q_T,$$
$$p \in L^2(0,T;H^1(\Omega)),\ \mathcal{B}(s_l) \in L^2(0,T;H^1_{\Gamma_l}(\Omega))$$

*such that for all* $\varphi \in \mathcal{C}^1(0,T;H^1_{\Gamma_l}(\Omega))$ *with* $\varphi(T) = 0$,

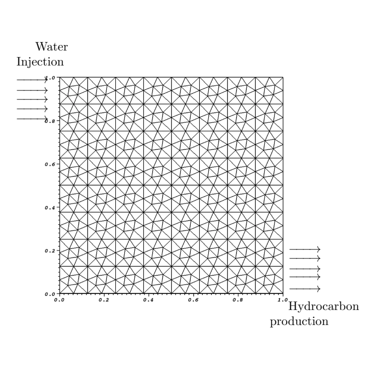
$$\begin{aligned}
&-\int_{Q_T} \phi\rho_\alpha(p_l)s_\alpha\partial_t\varphi \mathrm{d}x\mathrm{d}t - \int_\Omega \phi(x)\rho_\alpha(p^0_\alpha(x))s^0_\alpha(x)\varphi(0,x)\mathrm{d}x \\
&+\int_{Q_T} \rho_\alpha(p_l)M_l(s_\alpha)\boldsymbol{\Lambda}\nabla p_\alpha\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t - \int_{Q_T} \boldsymbol{\Lambda}M_\alpha(s_l)\rho^2_\alpha(p_l)\mathbf{g}\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t \\
&=\int_{Q_T} f_\alpha\varphi\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

## Initial and boundary conditions

Water-gas flow.
We simulate the waterflood method. The gas phase is slightly compressible, the water phase is incompressible.

$$k_l(s_l) = s_l^2,\ k_g(s_g) = s_g^2$$
$$k = 0.15 10^{-10}\text{m}^2,\ \ \Phi = 0.206,$$
$$\mu_l = 10^{-3}\ \text{Pa.s},\ \ \mu_g = 9\times 10^{-5}\ \text{Pa.s},$$
$$\rho_g(p_g) = \rho_r(1 + c_r(p_g - p_r)),\ \ \rho_r = 400\ \text{Kg}\,\text{m}^{-3},\ c_r = 0.1\text{bar}^{-1},\ p_r = 1\ \text{bar}$$
$$L_x = 1\text{m},\ L_y = 1\text{m}$$
$$P_c(s) = P_{max}(1-s),\ \text{with}\ P_{max} = 10^5\text{Pa}.$$

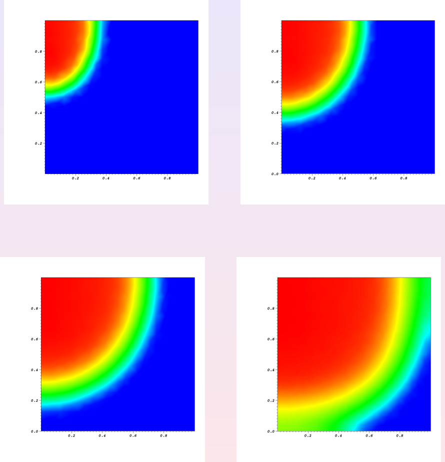
Initially $s_g(x,0) = 0.9$ and $p_g(x,0) = 1$bar in the whole domain.
Water is injected in the region $(x = 0, 0.8 \leq y \leq 1)$ with pressure $= 4$ bar.
Flow freely at atmospheric pressure in the region where $(x = 1, 0 \leq y \leq 0.2)$
the rest of the boundary is assumed to be impervious (zero fluxes are imposed).




Mesh with 896 triangles (2D Benchmark FVCA5)

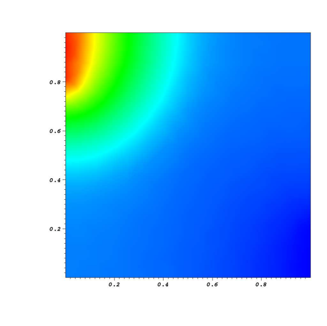
The mesh satisfies the orthogonal property. $\Lambda = k\,\mathrm{Id}$ and $k = 0.15 10^{-10} \mathrm{m}^2$.

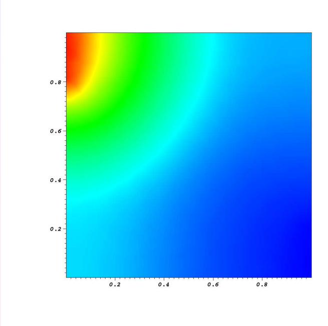
## Water field, T=2.4, 8, 16, 32



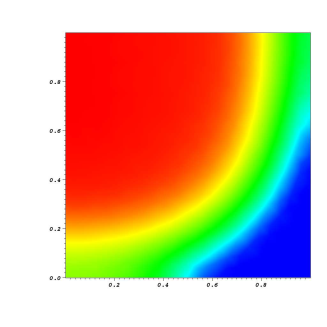
We show the diffusive effects of the capillary terms, notably the dissipation of front due to the parabolic operator.

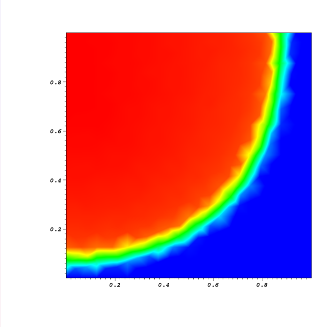
## Gas pressure field, $T = 16$ et $T = 32$



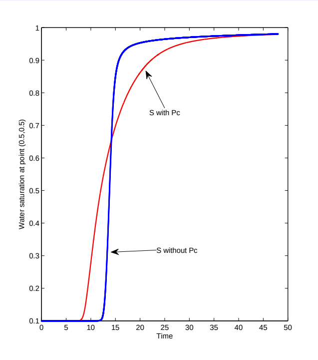
## Water field including capillary terms (left) and without capillary terms (right)

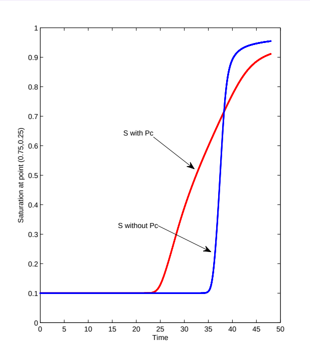



When capillary terms are neglected, the sharp front between the two fluids is then located and the shock wave maintains the sharp front during the simulation. We observe also that the water breakthrough time is shorter when capillarity terms are included.

## Evolution of the water saturation with respect to time at point (0.5,0.5) and $(0.75, 0.25)$




Comparison between flow including capillary effect or not.