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Hyperbolicity of two by two systems with two independent variables

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Abstract

We study the simplest system of partial differential equations: that is, two equations of first order partial differential equation with two independent variables with real analytic coefficients. We describe a necessary and sufficient condition for the Cauchy problem to the system to be C infinity well posed. The condition will be expressed by inclusion relations of the Newton polygons of some scalar functions attached to the system. In particular, we can give a characterization of the strongly hyperbolic systems which includes a fortiori symmetrizable systems.

1. Results.

Let us consider

$$Lu = \partial_t u - A(t,x)\partial_x u + B(t,x)u$$

where $t, x \in \mathbb{R}$ and $A(t,x)$ is a 2×2 matrix valued real analytic real valued function defined near the origin. We are concerned with the following Cauchy problem

$$\text{(C.P.)} \qquad \begin{cases} Lu = f \\ u(\hat{t}, x) = u_0(x). \end{cases}$$

We say that the Cauchy problem (C.P.) is C^∞ well posed in a neighborhood of the origin if there is a neighborhood W of the origin such that for any $(\hat{t},\hat{x}) \in W$ and any given $u_0(x) \in C^\infty(W \cap \{t = \hat{t}\})$ and $f \in C^\infty(W)$ the problem (C.P.) has a C^∞ solution in a neighborhood of $(\hat{t},\hat{x})$.

After a change of local coordinates around the origin leaving the lines t =const. invariant, we may assume that

$$A(t,x) = \begin{pmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{21}(t,x) & -a_{11}(t,x) \end{pmatrix}.$$

Let us denote $h(t,x) - \det A(t,x)$. It is well known that $h(t,x) \geq 0$ is necessary for the Cauchy problem (C.P.) to be well posed in a neighborhood of the origin

(Lax-Mizohata Theorem [3], [6]). Thus we assume $h(t,x) \geq 0$ in a neighborhood of the origin throughout the paper.

Let us set

$$c^\sharp = i(a_{12} - a_{21})/2, \quad a^\sharp(a_{12} + a_{21})/2 + ia_{11}, \quad D^\sharp = c^\sharp \partial_t a^\sharp - a^\sharp \partial_t c^\sharp.$$

Suppose that $h(t,x)$ does not vanish identically. Then $|a^\sharp|$ does not vanish identically because $h = |a^\sharp|^2 - |c^\sharp|^2$ and $h(t,x) \geq 0$ (see Lemma 3.1 below). From the Weierstrass preparation theorem $h(t,x)|a^\sharp(t,x)|^2$ is written as

$$h(t,x)|a^\sharp(t,x)|^2 = x^{2n}(t^{2r} + \phi_1(x)t^{2r-1} + \cdots + \phi_{2r}(x))E(t,x) \tag{1.1}$$

where $\phi_j(0) = 0$, $E(0,0) \neq 0$ and hence factorized as

$$h(t,x)|a^\sharp(t,x)|^2 = x^{2n} \prod_{j=1}^{2r} (t - t_j(x))E(t,x)$$

where $t_j(x)$ has a Puiseux expansion

$$t_j(x) = \sum_{k \geq 0} C_{jk}^{\pm} (\pm x)^{k/p_j}, \quad 0 < \pm x < \delta,$$

with some $p_j \in \mathbb{N}$. We set

$$\mathcal{F}_{\pm}(A) = \{\mathrm{Re} t_1(x), ..., \mathrm{Re} t_{2r}(x), \ \pm x > 0\}.$$

If $r = 0$ we put $\mathcal{F}_{\pm}(A) = \{0\}$.

Let $f(t,x)$ be real analytic near the origin and $\phi_{\pm} \in \mathcal{F}_{\pm}(A)$. We set

$$f_{\phi_{\pm}}(t,x) = f(t + \phi_{\pm}(x), x)$$

and define the Newton polygon $\Gamma(f_{\phi_{\pm}})$ of $f_{\phi_{\pm}}$ at $(0, \pm 0)$ as follows. For sufficiently small $|x|$, $\pm x > 0$ we have

$$f_{\phi_{\pm}}(t,x) = \sum_{i,j \geq 0} a_{ij}^{\pm} t^i (\pm x)^{j/p_{\pm}}$$

with some $p_{\pm} \in \mathbb{N}$ then set

$$\Gamma(f_{\phi_{\pm}}) = \text{convex hull of } \{ \bigcup_{a_{ij}^{\pm} \neq 0} (i, j/p_{\pm}) + \mathbb{R}_+^2 \}.$$

We define $\Gamma(f_{\phi_{\pm}}) = \emptyset$ if and only if f vanishes identically.

Theorem 1.1 *In order that the Cauchy problem* (C.P.) *is C^∞ well posed in a neighborhood of the origin, it is necessary and sufficient that*

$$\begin{aligned} \Gamma(t[D^\sharp + a^\sharp \mathrm{tr}(AB)]_\phi) &\subset \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi), \quad \forall \phi \in \mathcal{F}_{\pm}(A), \\ \Gamma(t[D^\sharp + a^\sharp \mathrm{tr}(A\bar{B})]_\phi) &\subset \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi), \quad \forall \phi \in \mathcal{F}_{\pm}(A) \end{aligned}$$

where $\bar{B}$ denotes the complex conjugate of B and $\mathrm{tr}A$ *denotes the trace of A.*

From Theorem 1.1 and [8, Theorem 1.1] it follows that the well posedness of the Cauchy problem (C.P.) is equivalent to the well posedness of the Cauchy problems for the following second order scalar operators:

$$\begin{aligned}&\partial_t^2 - h|a^\sharp|^2\partial_x^2 + (D^\sharp + a^\sharp \mathrm{tr}(AB))\partial_x,\\ &\partial_t^2 - h|a^\sharp|^2\partial_x^2 + (D^\sharp + a^\sharp \mathrm{tr}(A\bar{B}))\partial_x.\end{aligned}$$

Corollary 1.2 *Assume that $D^\sharp = 0$ or $|c^\sharp|^2 \le Ch$ with some $C > 0$ near the origin. Then for the Cauchy problem* (C.P.) *to be C^∞ well posed in a neighborhood of the origin, it is necessary and sufficient that*

$$\Gamma(t[\mathrm{tr}(AB)]_\phi) \subset \frac{1}{2}\Gamma(h_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A). \tag{1.2}$$

In particular, the Cauchy problem (C.P.) *is C^∞ well posed if $B = O$.*

If $D^\sharp = 0$ then both the conditions in Theorem 1.1 are reduced to

$$\Gamma(t[a^\sharp \mathrm{tr}(AB)]_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A). \tag{1.3}$$

By the definition, it is easy to check that the condition (1.2) is equivalent to (1.3). We now suppose that $|c^\sharp|^2 \le Ch$ with some $C > 0$ and $D^\sharp$ does not vanish identically. Then it is clear that

$$\Gamma(c^\sharp_\phi) \subset \frac{1}{2}\Gamma(h_\phi), \quad \phi \in \mathcal{F}_\pm(A). \tag{1.4}$$

Since $\Gamma(t[\partial_t c^\sharp]_\phi) = \Gamma(t\partial_t c^\sharp_\phi) \subset \Gamma_\pm(c^\sharp_\phi)$ we have

$$\begin{aligned}\Gamma(t[a^\sharp\partial_t c^\sharp]_\phi) &= \Gamma(a^\sharp_\phi) + \Gamma(t[\partial_t c^\sharp]_\phi) \subset \Gamma(a^\sharp_\phi) + \Gamma(c^\sharp_\phi)\\ \subset \frac{1}{2}\Gamma([|a^\sharp|^2]_\phi) + \frac{1}{2}\Gamma(h_\phi) &= \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi).\end{aligned}$$

A similar argument shows that $\Gamma(t[c^\sharp\partial_t a^\sharp]_\phi) \subset \Gamma([h|a^\sharp|^2]_\phi)/2$ and hence

$$\Gamma(tD^\sharp_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A). \tag{1.5}$$

Thus both the conditions in Theorem 1.1 are reduced to (1.3) and the rest of the proof is clear. □

REMARK 1.1: Let us consider a second order scalar equation with two independent variables:

$$Pv = \partial_t^2 v - a(t,x)\partial_x^2 v + b(t,x)\partial_x v = f.$$

With $u^I = \partial_x v$, $u^{II} = \partial_t v$ and $u = {}^t(u^I, u^{II})$ the equation is reduced to the following 2×2 system:

$$Lu = \partial_t u - \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}\partial_x u + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} u = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

It is clear that $a^\sharp = (1+a)/2$, $D^\sharp = i\partial_t a/2$ and $h = a$. Since $a^\sharp(0,0) \neq 0$, the set $\mathcal{F}_\pm(A)$ coincides with that defined in [8] for the scalar operator P. On the other hand from

$$\Gamma(t[\partial_t a]_\phi) = \Gamma(t\partial_t a_\phi) \subset \Gamma(a_\phi) \subset \frac{1}{2}\Gamma(a_\phi)$$

the condition (1.4) is verified. Thus the conditions in Theorem 1.1 are reduced to

$$\Gamma(t[\mathrm{tr}(AB)]_\phi) = \Gamma(tb_\phi) \subset \frac{1}{2}\Gamma(a_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A).$$

This is exactly the same condition obtained in [8, Theorem 1.1].

Corollary 1.3 *Assume that $h(t,x)$ vanishes identically. Then in order that the Cauchy problem* (C.P.) *is C^∞ well posed in a neighborhood of the origin, it is necessary and sufficient that*

$$D^\sharp + a^\sharp \mathrm{tr}(AB) \equiv 0, \quad D^\sharp + a^\sharp \mathrm{tr}(A\bar{B}) \equiv 0.$$

In the case h vanishes identically, a necessary and sufficient condition for the well posedness of (C.P.) was obtained in [4], [5] and in [10]. See also [2] and the references given there. We examine that the conditions given in Corollary 1.3 are equivalent to that obtained in [10]. Since $A^2 = O$ one can write

$$A = \begin{pmatrix} K\sigma\rho & K\sigma^2 \\ -K\rho^2 & -K\sigma\rho \end{pmatrix}$$

where σ and ρ are relatively prime. It is clear that $c^\sharp = iK(\sigma^2+\rho^2)/2$ and $a^\sharp = K(\sigma^2-\rho^2)/2 + iK\sigma\rho$. It is not difficult to check that $D^\sharp = K(\rho\partial_t\sigma - \sigma\partial_t\rho)a^\sharp$. Let

$$B = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}.$$

Since $\mathrm{tr}(AB) = K[b_1^2\sigma^2 - b_2^1\rho^2 + (b_1^1 - b_2^2)\sigma\rho]$ it follows that

$$D^\sharp + a^\sharp \mathrm{tr}(AB) = a^\sharp K[\rho\partial_t\sigma - \sigma\partial_t\rho + b_1^2\sigma^2 - b_2^1\rho^2 + (b_1^1 - b_2^2)\sigma\rho].$$

Thus the conditions given in Corollary 1.3 are equivalent to

$$\rho\partial_t\sigma - \sigma\partial_t\rho + b_1^2\sigma^2 - b_2^1\rho^2 + (b_1^1 - b_2^2)\sigma\rho = 0$$

which is exactly the Levi condition obtained in [10].

We turn to strong hyperbolicity. We say that L is strongly hyperbolic near the origin if the Cauchy problem (C.P.) is C^∞ well posed near the origin for all $B(t,x) \in C^\infty$.

Theorem 1.4 *For L to be strongly hyperbolic near the origin it is necessary and sufficient that*

$$\Gamma(tD^\sharp_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi), \quad \Gamma(t[a_{ij}]_\phi) \subset \frac{1}{2}\Gamma(h_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A).$$

Remark 1.2: Let
$$A=\begin{pmatrix} tx & t^2-x^2 \\ t^2-x^4 & -tx \end{pmatrix}.$$
Then the second condition of Theorem 1.4 is verified while the first condition is not. This example shows that in order that L is strongly hyperbolic the strong hyperbolicity of the second order scalar operators

$$\partial_t^2 - h(t,x)\partial_x^2 + a_{ij}(t,x)\partial_x$$

is necessary but not sufficient.

2. Examples.

In this section we give several examples to explain Theorem 1.1.

Example 1: We give an example of A with $h(t,x)>0$ outside $t=0$ for which no $B(t,x)$ could be taken so that (C.P.) is well posed (such an example was given in [4, Example 5] for the first time when h vanishes identically and was called "stably non hyperbolic" operator there). Let

$$A=\begin{pmatrix} x^2-t^4/2 & x^2+xt^2 \\ -x^2+xt^2 & -(x^2-t^4/2) \end{pmatrix}.$$

It is easy to see that

$$h=t^8/4,\ c^\sharp = ix^2,\ a^\sharp = xt^2+i(x^2-t^4/2),\ D^\sharp = 2ix^3t+2x^2t^3.$$

Suppose that $B(t,x)=(b_{ij}(t,x))$ is given. It is easy to check that $a^\sharp \mathrm{tr}(AB)$ has the form
$$C_{40}x^4+C_{31}x^3t^2+C_{24}x^2t^4+C_{16}xt^6+C_{08}t^8$$
where $C_{ij}(t,x)$ is a linear combination of $b_{ij}(t,x)$. On the other hand we have

$$h|a^\sharp|^2 = t^8(x^4+t^8/4)/4 = x^4t^8/4+t^{16}/16.$$

Taking $\phi=0$ we easily see that

$$\Gamma(t[D^\sharp + a^\sharp \mathrm{tr}(AB)]) \not\subset \frac{1}{2}\Gamma([h|a^\sharp|^2])$$

for any B because $D^\sharp + a^\sharp\mathrm{tr}(AB)$ has the form

$$2ix^3t+2x^2t^3+C_{40}x^4+C_{31}x^3t^2+C_{24}x^2t^4+C_{16}xt^6+C_{08}t^8$$

and no B cancels $2ix^3t$.

Example 2: Symmetric systems

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}, \quad a_{12}=a_{21}.$$

In this case we have

$$c^\sharp = 0, \quad D^\sharp = 0, \quad |a_{11}| \le |a^\sharp|, \quad |a_{12}| = |a_{21}| \le |a^\sharp|, \quad h = |a^\sharp|^2$$

and hence $|\mathrm{tr}(AB)|^2 \le Ch$ for every smooth B. In particular, this shows that (1.2) is verified for every B and hence Corollary 1.2 is available.

Example 3: We give two examples for which (1.2) is a necessary and sufficient condition for the C^∞ well posedness of (C.P.) while the assumption in Corollary 1.2 is not necessarily verified. Let

$$A = \begin{pmatrix} f(t,x) & f(t,x) - g(t,x) \\ -f(t,x) - g(t,x) & -f(t,x) \end{pmatrix}$$

then it is clear that

$$h = g^2, \ c^\sharp = if, \ a^\sharp = -g + if, \ D^\sharp = i(g\partial_t f - f\partial_t g).$$

We take f, g so that $C|f| \ge |g|$ and $\Gamma(g_\phi) \ne \Gamma(f_\phi)$ for some ϕ. Since

$$\frac{1}{2}\Gamma(h_\phi) = \Gamma(g_\phi) \subset \Gamma(f_\phi)\Gamma_\pm(c^\sharp_\phi)$$

the assumption in Corollary 1.2 does not hold. On the other hand the same argument employed in the proof of Corollary 1.2 shows that (1.5) is verified. Then the condition (1.2) is necessary and sufficient for C^∞ well posedness of (C.P.).

Let

$$A = \begin{pmatrix} d(t,x) & a(t,x) \\ b(t,x) & -d(t,x) \end{pmatrix}$$

and assume that $h = d^2 + ab \ge \delta d^2$ in a neighborhood of the origin with some positive constant $\delta > 0$. Note that $a^\sharp(a+b)/2 + id$ and

$$D^\sharp = \frac{i}{2}(a\partial_t b - b\partial_t a) - \frac{1}{2}(a\partial_t d - d\partial_t a - b\partial_t d + d\partial_t b).$$

It is easy to see that

$$a^2b^2, \ a^2d^2, \ b^2d^2 \le Ch|a^\sharp|^2$$

in a neighborhood of the origin with some $C > 0$ because

$$a^2 + b^2 + |ab| + d^2 \le C|a^\sharp|^2, \quad d^2 + |ab| \le Ch$$

with some $C > 0$ by the assumption. Then repeating the same argument as in Corollary 1.2 we conclude that (1.5) holds and hence (1.2) is again necessary and sufficient for C^∞ well posedness (a related result can be found in [9]).

Example 4: Uniformly diagonalizable 2×2 hyperbolic systems with two independent variables. Assume that for every (t,x) near the origin there is a $U(t,x)$ such that

$$U(t,x)^{-1}A(t,x)U(t,x)$$

becomes diagonal matrix and $\|U(t,x)\|$, $\|U(t,x)^{-1}\| \le C$ with some $C>0$ which is independent of (t,x) where $\|U\|^2 = \mathrm{tr}(U^tU)$. Let us denote $A^U = U^{-1}AU = (a^U_{ij})$. Note that (see (3.1) below)

$$|(a^U)^\sharp|^2 = \frac{h}{2} + \frac{1}{4}\|A^U\|^2 \ge \frac{h}{2} + \frac{1}{4}C^{-4}\|A\|^2 \ge C^{-4}|a^\sharp|^2.$$

On the other hand since A^U is diagonal and $\mathrm{tr}(A^U)=0$ we have $|(a^U)^\sharp|^2 = h$. This shows that

$$|a^\sharp|^2 \le C^4 h.$$

Thus we have $|c^\sharp|^2 \le 2|a^\sharp|^2 \le C'h$ and hence (1.5) holds as we have seen in the proof of Corollary 1.2. On the other hand $|a_{ij}|^2 \le 4|a^\sharp|^2 \le C''h$ proves

$$\Gamma(t[a_{ij}]_\phi) \subset \Gamma([a_{ij}]_\phi) \subset \frac{1}{2}\Gamma(h_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A).$$

Thus the conditions in Theorem 1.1 are satisfied.

Example 5: A non symmetrizable strongly hyperbolic system. Let

$$A = \psi(t,x)\begin{pmatrix} 0 & 1 \\ t^2 & 0 \end{pmatrix}.$$

In this case we have

$$a^\sharp = \psi(1+t^2)/2,\ c^\sharp = i\psi(1-t^2)/2,\ D^\sharp = i\psi^2 t,\ \ h = t^2\psi^2.$$

We note that

$$\Gamma(tf_\phi) = \Gamma(t) + \Gamma_\pm(f_\phi) \subset \Gamma([tf]_\phi)$$

because $\Gamma(t) \subset \Gamma(t_\phi)$. Then remarking $|ta_{ij}|^2 \le Ch$, $|tD^\sharp|^2 \le Ch|a^\sharp|^2$ we see that

$$\Gamma(t[a_{ij}]_\phi) \subset \Gamma([ta_{ij}]_\phi) \subset \frac{1}{2}\Gamma(h_\phi), \quad \Gamma(tD^\sharp_\phi) \subset \Gamma([tD^\sharp]_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp|^2]_\phi)$$

and hence the conditions in Theorem 1.4 are verified.

Example 6: Some not strongly hyperbolic systems. Let

$$A = \psi(t,x)\begin{pmatrix} 0 & 1 \\ t^4 & 0 \end{pmatrix}.$$

For this A we have

$$a^\sharp = \psi(1+t^4)/2,\ c^\sharp = i\psi(1-t^4)/2,\ D^\sharp = 2it^3\psi^2,\ \ h = t^4\psi^2.$$

Since $\Gamma(tD^\sharp_\phi) \subset \Gamma([h|a^\sharp|^2]_\phi)/2$ is clear the condition (1.2) is necessary and sufficient for the C^∞ well posedness of (C.P.). Denoting

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

the condition (1.2) is reduced to $\Gamma(t[\psi b_{21}]_\phi) \subset \Gamma(h_\phi)/2$. This is equivalent to $b_{21}(0,x) = 0$.

We now let

$$A = \psi(t,x)\begin{pmatrix} a(x) & a(x)+b(x) \\ -a(x)+b(x) & -a(x) \end{pmatrix}.$$

It is clear that $h = \psi^2 b(x)^2$, $D^\sharp = 0$ and hence (1.2) is necessary and sufficient for the C^∞ well posedness by Corollary 1.2. Since

$$\mathrm{tr}(AB) = a(b_{11} + b_{21} - b_{12} - b_{22})\psi + b(b_{21} + b_{12})\psi$$

the condition (1.2) is reduced to

$$\Gamma(t[a(b_{11} + b_{21} - b_{12} - b_{22})]_\phi) \subset \Gamma(b_\phi), \quad \forall \phi \in \mathcal{F}_\pm(A).$$

Since b is independent of t this is further reduced to

$$\Gamma(a(b_{11} + b_{21} - b_{12} - b_{22})) \subset \Gamma(b).$$

3. Sketch of the proof of Theorem 1.1.

We now sketch our strategy to prove Theorem 1.1. Let

$$T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and study $T^{-1}LT$ which turns out to be $\partial_t - A^\sharp\partial_x + B^\sharp$ where

$$\begin{aligned} A^\sharp(t,x) \quad & \begin{pmatrix} i(a_{12}-a_{21})/2 & (a_{12}+a_{21})/2 + ia_{11} \\ (a_{12}+a_{21})/2 - ia_{11} & -i(a_{12}-a_{21})/2 \end{pmatrix} \\ & = \begin{pmatrix} a^\sharp_{11}(t,x) & a^\sharp_{12}(t,x) \\ a^\sharp_{21}(t,x) & -a^\sharp_{11}(t,x) \end{pmatrix}. \end{aligned}$$

Note that $a^\sharp = a^\sharp_{12}$, $\overline{a^\sharp} = a^\sharp_{21}$, $c^\sharp = a^\sharp_{11}$ and

$$a^\sharp_{12}a^\sharp_{21}h/2 + \mathrm{tr}(A^tA)/4, \quad h = (a^\sharp_{11})^2 + a^\sharp_{12}a^\sharp_{21} - |a^\sharp_{11}|^2 + |a^\sharp_{12}|^2. \tag{3.1}$$

In particular, the first identity shows that, with $A^T = T^{-1}A(x)T = (a^T_{ij}(x))$

$$|(a^T_{12})^\sharp| = |a^\sharp_{12}|$$

for every orthogonal matrix T. Moreover since $h \geq 0$, for any non singular S, there is $C = C(S) > 0$ such that

$$C^{-1}|a^\sharp_{12}| \leq |(a^S_{12})^\sharp| \leq C|a^\sharp_{12}|. \tag{3.2}$$

It is clear from $h(t,x) \geq 0$ that

Lemma 3.1 *We have*

$$|a^\sharp_{12}| = |a^\sharp_{21}| \geq |a^\sharp_{11}|, \quad 4|a^\sharp_{12}|^2 \geq \mathrm{tr}(A^t A) = \sum_{i,j=1}^{2} a^2_{ij}(x), \quad |a^\sharp_{12}|^2 \geq h.$$

In particular, $a^\sharp_{12}(t,x) = 0$ is equivalent to $A(t,x) = O$.

Let us set

$$M = \partial_t + A^\sharp \partial_x + C + {}^{co}B^\sharp - A^\sharp_x$$

and study $L^\sharp M$ where ${}^{co}B^\sharp$ denotes the cofactor matrix of $B^\sharp$ and $A^\sharp_x = \partial_x A^\sharp$. This turns out

$$\begin{aligned} L^\sharp M &= \partial_t^2 - h\partial_x^2 + (A^\sharp_t - A^\sharp C + \mathrm{tr}(AB)I)\partial_x \\ &\quad +(B^\sharp + {}^{co}B^\sharp + C - A^\sharp_x)\partial_t + L^\sharp(C + {}^{co}B^\sharp - A^\sharp_x). \end{aligned}$$

In fact taking $B^\sharp A^\sharp - A^{\sharp co} B^\sharp = B^\sharp A^\sharp + {}^{co}(B^\sharp A^\sharp) = \mathrm{tr}(A^\sharp B^\sharp)I = \mathrm{tr}(AB)I$ into account the identity is easily seen. With $C = (c_{ij})$ the coefficient $A^\sharp_t - A^\sharp C + \mathrm{tr}(AB)$ becomes

$$\begin{pmatrix} \partial_t a^\sharp_{11} - a^\sharp_{11} c_{11} - a^\sharp_{12} c_{21} + \mathrm{tr}(AB) & \partial_t a^\sharp_{12} - a^\sharp_{11} c_{12} - a^\sharp_{12} c_{22} \\ \partial_t a^\sharp_{21} + a^\sharp_{11} c_{21} - a^\sharp_{21} c_{11} & -\partial_t a^\sharp_{11} + a^\sharp_{11} c_{22} - a^\sharp_{21} c_{12} + \mathrm{tr}(AB) \end{pmatrix}. \tag{3.3}$$

We determine c_{ij} by

$$\partial_t a^\sharp_{12} - a^\sharp_{11} c_{12} - a^\sharp_{12} c_{22} = 0, \quad \partial_t a^\sharp_{21} + a^\sharp_{11} c_{21} - a^\sharp_{21} c_{11} = 0 \tag{3.4}$$

so that (3.3) will be diagonal. Then (3.3) becomes

$$\begin{pmatrix} Y(t,x)/a^\sharp_{21} - c_{21} h(t,x)/a^\sharp_{21} & 0 \\ 0 & Z(t,x)/a^\sharp_{12} - c_{12} h(t,x)/a^\sharp_{12} \end{pmatrix}$$

where

$$\begin{aligned} Y(t,x) &= a^\sharp_{21} \partial_t a^\sharp_{11} - a^\sharp_{11} \partial_t a^\sharp_{21} + a^\sharp_{21} \mathrm{tr}(AB), \\ Z(t,x) &= -a^\sharp_{12} \partial_t a^\sharp_{11} + a^\sharp_{11} \partial_t a^\sharp_{12} + a^\sharp_{12} \mathrm{tr}(AB). \end{aligned}$$

We take $c_{12} = 0$ and $c_{21} = 0$ so that (3.4) implies

$$c_{11} = \partial_t a^\sharp_{21}/a^\sharp_{21}, \quad c_{22} = \partial_t a^\sharp_{12}/a^\sharp_{12}.$$

We summarize:

Lemma 3.2 *Let*

$$M = \partial_t + A^\sharp \partial_x - A^\sharp_x + {}^{co}B^\sharp + C$$

with

$$C = \mathrm{diag}\left(\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}, \frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}\right).$$

Then one can write

$$L^\sharp M = \partial_t^2 - h\partial_x^2 + Q\partial_x + R\partial_t + S$$

where

$$Q = \mathrm{diag}\left(\frac{Y(t,x)}{a^\sharp_{21}(t,x)}, \frac{Z(t,x)}{a^\sharp_{12}(t,x)}\right), \; R = C - A^\sharp_x + B^\sharp + {}^{co}B^\sharp$$

and $S = L^\sharp(C) + L^\sharp({}^{co}B^\sharp - A^\sharp_x)$.

Here we remark that from $a^\sharp_{12} = \overline{a^\sharp_{21}}$ and $\overline{a^\sharp_{11}} - a^\sharp_{11}$ it follows that

$$\bar{Y} = D^\sharp + a^\sharp \mathrm{tr}(A\bar{B}), \quad Z = D^\sharp + a^\sharp \mathrm{tr}(AB) \tag{3.5}$$

and hence

$$\begin{pmatrix} \bar{a}^\sharp & 0 \\ 0 & a^\sharp \end{pmatrix} Q = \begin{pmatrix} \overline{D^\sharp + a^\sharp \mathrm{tr}(A\bar{B})} & 0 \\ 0 & D^\sharp + a^\sharp \mathrm{tr}(AB) \end{pmatrix}.$$

We also note that $L^\sharp(C)$ has the form

$$\begin{aligned} L^\sharp(C) &= \mathrm{diag}\left(\partial_t\left(\partial_t a^\sharp_{21}/a^\sharp_{21}\right),\ \partial_t\left(\partial_t a^\sharp_{12}/a^\sharp_{12}\right)\right) \\ &\quad + \begin{pmatrix} a^\sharp_{11}\partial_x\left(\partial_t a^\sharp_{21}/a^\sharp_{21}\right) & a^\sharp_{12}\partial_x\left(\partial_t a^\sharp_{12}/a^\sharp_{12}\right) \\ a^\sharp_{21}\partial_x\left(\partial_t a^\sharp_{21}/a^\sharp_{21}\right) & -a^\sharp_{11}\partial_x\left(\partial_t a^\sharp_{12}/a^\sharp_{12}\right) \end{pmatrix} + B^\sharp C. \end{aligned}$$

We next get

Lemma 3.3 *Let*

$$M = \partial_t + A^\sharp\partial_x + A^\sharp_x + {}^{co}B^\sharp + \tilde{C}$$

with

$$\tilde{C} = -\mathrm{diag}\left(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}, \frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}\right).$$

Then we have

$$ML^\sharp = \partial_t^2 - h\partial_x^2 - h_x\partial_x + \tilde{Q}\partial_x + \tilde{R}\partial_t + \tilde{S}$$

where

$$\tilde{Q} = \mathrm{diag}\left(\frac{Z}{a^\sharp_{12}}, \frac{Y}{a^\sharp_{21}}\right),\ \tilde{R} = \tilde{C} + A^\sharp_x + B^\sharp + {}^{co}B^\sharp,\ \tilde{S} = M(B^\sharp).$$

Note that $h_x = A^\sharp_x A^\sharp + A^\sharp A^\sharp_x$ and $A^\sharp B^\sharp - {}^{co}B^\sharp A^\sharp = \mathrm{tr}(A^\sharp B^\sharp)I = \mathrm{tr}(AB)I$. Then to prove the assertion it is enough to repeat similar computations as in the proof of Lemma 3.2. □

Here we note that

$$\begin{pmatrix} a^\sharp & 0 \\ 0 & \bar{a}^\sharp \end{pmatrix} \tilde{Q} = \begin{pmatrix} D^\sharp + a^\sharp \mathrm{tr}(AB) & 0 \\ 0 & \overline{D^\sharp + a^\sharp \mathrm{tr}(A\bar{B})} \end{pmatrix}.$$

To prove the necessity of the condition we construct an asymptotic solution U_λ, depending on a large parameter λ, to the Cauchy problem for $L^\sharp_\lambda$, which results from $L^\sharp$ by a dilation of local coordinates such as $(t,x) \mapsto (\lambda^{-p}t, \lambda^{-q}x)$ with p, $q \in \mathbb{Q}_+$. We look for U_λ in the form $U_\lambda = M_\lambda V_\lambda$ where M is given in Lemma 3.2. That is, we construct an asymptotic solution V_λ to $L^\sharp_\lambda M_\lambda V_\lambda \sim 0$ which violates an a priori estimate derived from well posed assumption of the Cauchy problem (C.P.) for L. Here with $L^\sharp M = \partial_t^2 - h\partial_x^2 + Q\partial_x + R\partial_t + S$ we have

$$L^\sharp_\lambda M_\lambda = \lambda^{2p}\partial_t^2 - \lambda^{2q}h_\lambda\partial_x^2 + \lambda^q Q_\lambda\partial_x + \lambda^p R_\lambda\partial_t + S_\lambda.$$

We must be careful when we treat the lower order terms because Q, R and S are, in general, no more smooth at the origin because of our choice of C (see the form $L^\sharp(C)$ in (3.6) for example). Then singularities of Q, R, S at the origin contribute as a positive power of λ, in the resulting functions Q_λ, R_λ and S_λ. A main point in the proof of the necessity is that, with this choice of C, the existence of a desired asymptotic solution depends upon the positive power of λ in Q_λ, that is whether $\mathrm{diag}(\bar{a^\sharp}, a^\sharp)Q$ verifies the condition in Theorem 1.1 or not and independent of the yielded positive powers of λ in R_λ and S_λ.

Since the existence of analytic solutions with analytic data is assured by the Cauchy -Kowalewski theorem, applying the usual limiting arguments, to prove the sufficiency of the condition, it is enough to derive an a priori estimate of analytic solution to $L^\sharp u = f$. Since u verifies

$$ML^\sharp u = (\partial_t^2 - h\partial_x^2 + (\tilde{Q} - h_x)\partial_x + \tilde{R}\partial_t + \tilde{S})u = Mf$$

we use this equation to get an a priori estimate, where M is given in Lemma refoneseven. One of main ideas is that we regard the zeros of $h|a^\sharp|^2$ as characteristics. That is, we study not only the zeros of h but also those of $a^\sharp$ which tells us precise behaviors of $\sqrt{h}|a^\sharp|$ near the origin. According to the behavior of $\sqrt{h}|a^\sharp|$ we divide a neighborhood of the origin into several subregions and we derive a weighted a priori estimate in each subregion, where the weight is chosen taking the behavior of $\sqrt{h}|a^\sharp|$ into account. A key observation to get a weighted a priori estimate is that we can obtain a weighted estimate even when $\tilde{R}$ and $\tilde{S}$ are not smooth. More precisely if $t(x)$ is a zero of $\sqrt{h}|a^\sharp|$ with respect to t and $\tilde{R} = O((t - \mathrm{Re}t(x))^{-1})$, $\tilde{S} = O((t - \mathrm{Re}t(x))^{-2})$ as $t - \mathrm{Re}t(x) \to 0$, then we can obtain a weighted a priori estimate with weights $(t - \mathrm{Re}t(x))^N$, $N \in \mathbb{Z}$ in a subregion mentioned above if $\mathrm{diag}(a^\sharp, \bar{a^\sharp})\tilde{Q}$ verifies the condition in Theorem 1.1.

Combining a priori estimate in each subregion thus obtained, we get a priori estimate in a full neighborhood of the origin.

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