A SYMPLECTIC RESOLUTION FOR THE BINARY TETRAHEDRAL GROUP

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Abstract. We describe an explicit symplectic resolution for the quotient singularity arising from the four-dimensional symplectic representation of the binary tetrahedral group.

Let G be a finite group with a complex symplectic representation V. The symplectic form σ on V descends to a symplectic form $\bar{\sigma}$ on the open regular part of V/G. A proper morphism $f : Y \to V/G$ is a symplectic resolution if Y is smooth and if $f^*\bar{\sigma}$ extends to a symplectic form on Y. It turns out that symplectic resolutions of quotient singularities are a rare phenomenon. By a theorem of Verbitsky [16], a necessary condition for the existence of a symplectic resolution is that G be generated by symplectic reflections, i.e. by elements whose fix locus on V is a linear subspace of codimension 2. Given an arbitrary complex representation V_0 of a finite group G, we obtain a symplectic representation on $V_0 \oplus V_0^*$, where V_0^* denotes the contragradient representation of V_0. In this case, Verbitsky's theorem specialises to an earlier theorem of Kaledin [10]: For $V_0 \oplus V_0^*/G$ to admit a symplectic resolution, the action of G on V_0 should be generated by complex reflections, in other words, V_0/G should be smooth. The complex reflection groups have been classified by Shephard and Todd [15], the symplectic reflection groups by Cohen [3]. The list of Shephard and Todd contains as a sublist the finite Coxeter groups.

The question which of these groups $G \subset \mathrm{Sp}(V)$ admits a symplectic resolution for V/G has been solved for the Coxeter groups by Ginzburg and Kaledin [4] and for arbitrary complex reflection groups most recently by Bellamy [1]. His result is as follows:

Theorem 1. (Bellamy) — *If $G \subset \mathrm{GL}(V_0)$ is a finite complex reflection group, then $V_0 \oplus V_0^*/G$ admits a symplectic resolution if and only if (G, V_0) belongs to the following cases:*

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1. $(S_n, \mathfrak{h})$, *where the symmetric group* S_n *acts by permutations on the hyperplane* $\mathfrak{h} = \{x \in \mathbb{C}^n \mid \sum_i x_i = 0\}$.
2. $((\mathbb{Z}/m)^n \rtimes S_n, \mathbb{C}^n)$, *the action being given by multiplication with* m*-th roots of unity and permutations of the coordinates.*
3. (T, S_1), *where* S_1 *denotes a two-dimensional representation of the binary tetrahedral group* T *(see below).*

However, the technique of Ginzburg, Kaledin and Bellamy does not provide resolutions beyond the statement of existence. Case 1 corresponds to Coxeter groups of type A and Case 2 with $m = 2$ to Coxeter groups of type B. It is well-known that symplectic resolutions of $\mathfrak{h} \oplus \mathfrak{h}^*/S_n$ and $\mathbb{C}^n \oplus \mathbb{C}^n/(\mathbb{Z}/m)^n \rtimes S_n \cong \mathrm{Sym}^n(\mathbb{C}^2/(\mathbb{Z}/m))$ are given as follows:

For a smooth surface Y the Hilbert scheme $\mathrm{Hilb}^n(Y)$ of generalised n-tuples of points on Y provides a crepant resolution $\mathrm{Hilb}^n(Y) \to \mathrm{Sym}^n(Y)$. Applied to a minimal resolution of the A_{m-1}-singularity $\mathbb{C}^2/G$, $G \cong \mathbb{Z}/m$, this construction yields a small resolution $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/G}) \to \mathrm{Sym}^n(\widetilde{\mathbb{C}^2/G}) \to \mathrm{Sym}^n(\mathbb{C}^2/G)$. Similarly, $(\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$ is the fibre over the origin of the barycentric map $\mathrm{Sym}^n(\mathbb{C}^2) \to \mathbb{C}^2$. Thus $(\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$ is resolved symplectically by the null-fibre of the morphism $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2) \to \mathbb{C}^2$.

It is the purpose of this note to describe an explicit symplectic resolution for the binary tetrahedral group.

1. The binary tetrahedral group

Let $T_0 \subset \mathrm{SO}(3)$ denote the symmetry group of a regular tetrahedron. The preimage of T_0 under the standard homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is the binary tetrahedral group T. As an abstract group, T is the semidirect product of the quaternion group $Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ and the cyclic group $\mathbb{Z}/3$. As a subgroup of $\mathrm{SU}(2)$ it is generated by the elements

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \tau = -\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

The binary tetrahedral group has 7 irreducible complex representations: A three-dimensional one arising from the quotient $T \to T_0 \subset \mathrm{SO}_3$, three one-dimensional representations $\mathbb{C}_j$ arising from the quotient $T \to \mathbb{Z}/3$ with τ acting by $e^{2\pi i j/3}$, and three two-dimensional representations S_0, S_1 and S_2. Here S_0 denotes the standard representation of T arising from the embedding $T \subset \mathrm{SU}_2$. This representation is symplectic, its quotient S_0/T being the well-known Klein-DuVal singularity of type E_6. The two other representations can be written as $S_j = S_0 \otimes \mathbb{C}_j$, $j = 1, 2$. They are dual to each other. It is as the subgroup of $\subset \mathrm{GL}(S_1)$ that T appears in the list of Shephard and Todd under the label "No. 4". The diagonal action of T on $S_1 \oplus S_2$ provides the embedding of T to Sp_4 that is of interest in our context.

Whereas the action of T on S_0 is symplectic, the action of T on S_1 and S_2 is generated by complex reflections of order 3. Overall, there are 8 elements of order 3 in T or rather 4 pairs of inverse elements, forming 2 conjugacy classes. To these correspond 4 lines in S_1 of points with nontrivial isotropy groups. Let $C_1 \subset S_1$ and $C_2 \subset S_2$ denote the union of these lines in each case. Then $C_1 \times S_2$ and $S_1 \times C_2$ are invariant divisors in $S_1 \oplus S_2$. However, the defining equations are invariant only up to a scalar. Consequently, their images W_1 and W_2 in the quotient $Z = S_1 \oplus S_2/T$ are Weil divisors but not Cartier. The reduced singular locus $\mathrm{sing}(Z)$ is irreducible and off the origin a transversal A_2 singularity. It forms one component of the intersection $W_1 \cap W_2$.

For $j = 1, 2$, let $\alpha_j : Z'_j \to Z$ denote the blow-up along W_j. Next, let W'_j be the reduced singular of locus Z'_j, and let $\beta_j : Z''_j \to Z'_j$ denote the blow-up along W'_j.

Theorem 2. — *The morphisms $\sigma_j = \alpha_j \beta_j : Z''_j \to Z$, $j = 1, 2$, are symplectic resolutions.*

Proof. As all data are explicit, the assertion can be checked by brute calculation. To cope with the computational complexity we use the free computer algebra system SINGULAR[1] [5]. It suffices to treat one of the two cases of the theorem. We indicate the basic steps for $j = 2$. In order to improve the readability of the formulae we write $q = \sqrt{-3}$.

Let $\mathbb{C}[x_1, x_2, x_3, x_4]$ denote the ring of polynomial functions on $S_1 \oplus S_2$. The invariant subring $\mathbb{C}[x_1, x_2, x_3, x_4]^T$ is generated by eight elements, listed in table 1. The kernel I of the corresponding ring homomorphism

$$\mathbb{C}[z_1, \ldots, z_8] \to \mathbb{C}[x_1, x_2, x_3, x_4]^T$$

is generated by nine elements, listed in table 2. The curve C_2 is given by the semiinvariant $x_3^4 + 2qx_3^2x_4^2 + x_4^4$. In order to keep the calculation as simple as possible, the following observation is crucial: Modulo I, the Weil divisor W_2 can be described by 6 equations, listed in table 3. This leads to a comparatively 'small' embedding $Z'_2 \to \mathbb{P}^5_Z$ of Z-varieties. Off the origin, the effect of blowing-up of W_2 is easy to understand even without any calculation:

[1]A documented SINGULAR file containing all the calculations is available from the authors upon request.

Table 1: generators for the invariant subring $\mathbb{C}[x_1,x_2,x_3,x_4]^T$:

$$\begin{aligned}
z_1 &= x_1x_3+x_2x_4, & z_4 &= x_2x_3^3-qx_1x_3^2x_4+qx_2x_3x_4^2-x_1x_4^3,\\
z_2 &= x_3^4-2qx_3^2x_4^2+x_4^4, & z_5 &= x_2^3x_3-qx_1^2x_2x_3+qx_1x_2^2x_4-x_1^3x_4,\\
z_3 &= x_1^4+2qx_1^2x_2^2+x_2^4, & z_6 &= x_1^5x_2-x_1x_2^5,\\
z_7 &= x_3^5x_4-x_3x_4^5 & z_8 &= x_1x_2^2x_3^3-x_2^3x_3^2x_4-x_1^3x_3x_4^2+x_1^2x_2x_4^3.
\end{aligned}$$

Table 2: generators for $I=\ker(\mathbb{C}[z_1,\dots,z_8]\to\mathbb{C}[x_1,\dots,x_4]^T)$.

$$\begin{aligned}
&qz_1^3z_5-z_1z_3z_4-2z_2z_6-z_5z_8, && z_1z_5^2+2z_4z_6+z_3z_8,\\
&qz_1^3z_4+z_1z_2z_5-2z_3z_7-z_4z_8, && z_1z_4^2-2z_5z_7-z_2z_8,\\
&-z_1^4+z_2z_3-z_4z_5-3qz_1z_8, && qz_1^2z_3z_5-2z_1^3z_6-z_3^2z_4+z_5^3-6qz_6z_8,\\
&z_1^2z_4z_5+qz_1^3z_8+4z_6z_7-z_8^2, && qz_1^2z_2z_4-2z_1^3z_7-z_4^3+z_2^2z_5-6qz_7z_8,\\
&4z_1^2z_4z_5+qz_3z_4^2-qz_2z_5^2+4z_6z_7+8z_8^2
\end{aligned}$$

Table 3: generators for the ideal of the Weil divisor $W_2\subset Z$.

$$\begin{aligned}
b_1 &= z_3z_7+2z_4z_8, & b_2 &= z_2z_4+2qz_1z_7, & b_3 &= z_2z_3-4qz_1z_8,\\
b_4 &= z_2^3+12qz_7^2, & b_5 &= z_1z_2^2-6z_4z_7, & b_6 &= z_1^2z_2-qz_4^2.
\end{aligned}$$

the action of the quaternion normal subgroup $Q_8\subset T$ on $S_1\oplus S_2\setminus\{0\}$ is free. The action of $\mathbb{Z}/3=T/Q_8$ on $S_1\oplus S_2/Q_8$ produces transversal A_2-singularities along a smooth two-dimensional subvariety. Blowing-up along W_1 or W_2 is a partial resolution: it introduces a $\mathbb{P}^1$ fibre over each singular point, and the total space contains a transversal A_1-singularity.

The homogeneous ideal $I_2'\subset\mathbb{C}[z_1,\dots,z_8,b_1,\dots,b_6]$ that describes the subvariety $Z_2'\subset\mathbb{P}^5_Z$ is generated by I and 39 additional polynomials. In order to understand the nature of the singularities of Z_2' we consider the six affine charts $U_\ell=\{b_\ell=1\}$. The result can be summarised like this: The singular locus of Z_2' is completely contained in $U_2\cup U_3$, so only these charts are relevant for the discussion of the second blow-up. In fact, the corresponding affine coordinate rings have the following description:

$$R_2=\mathbb{C}[z_1,b_3,b_4,b_5,b_6]/(b_5b_6-2qz_1)^2+b_4(3qb_3-b_6^3)$$

is a transversal A_1-singularity.

$$R_3=\mathbb{C}[z_1,z_3,z_5,z_6,b_1,b_2,b_6]/J,$$

where J is generated by five elements, listed in table 4. Inspection of these generators shows that $\mathrm{Spec}(R_2)$ is isomorphic to the singularity $(\mathfrak{h}_3\oplus\mathfrak{h}_3^*)/S_3$, the symplectic singularity of Coxeter type A_2 that appears as case 1 in

Table 4: generators for the ideal sheaf J of $Z_2' \subset \mathbb{C}^7$ in the third chart:

$$
\begin{array}{ll}
4z_1b_1 + qz_3b_2 + z_5b_6, & z_1z_5 + z_3b_1 + qz_6b_6, \\
z_1^2b_6 - z_3b_6^2 - 4qb_1^2 - 3z_5b_2, & z_1^2z_3 - z_3^2b_6 - qz_5^2 - 12z_6b_1, \\
z_1^3 - z_1z_3b_6 + qz_5b_1 + 3qz_6b_2 &
\end{array}
$$

Bellamy's theorem. It is well-known that blowing up the singular locus yields a small resolution. For arbitrary n, this is a theorem of Haiman [6, Prop. 2.6], in our case it is easier to do it directly. Thus blowing-up the reduced singular locus of Z_2' produces a smooth resolution $Z_2'' \to Z$.

It remains to check that the morphism $\alpha_2 : Z_2 \to Z$ is semi-small. For this it suffices to verify that the fibre $E = (\alpha_2^{-1}(0))_{\mathrm{red}}$ over the origin is two-dimensional and not contained in the singular locus of Z_2'. Indeed, the computer calculation shows that $E \subset \mathbb{P}^5$ is given by the equations b_1, b_3b_5, b_3b_4, $b_5^2 - b_4b_6$ and hence is the union of two irreducible surfaces. The singular locus of Z_2' is irreducible and two-dimensional and dominates the singular locus of Z. Thus the second requirement is fulfilled, too. □

2. The equivariant Hilbert scheme

Though the description of the resolution is simple and straight the method of proof is less satisfying. It is based on explicit calculation that given the complexity of the singularity we were able to carry out only by means of appropriate software. Remark that even for the classical ADE-singularities arising from finite subgroups $G \subset \mathrm{SU}(2)$ the actual resolutions of $\mathbb{C}^2/G$ could only be described by explicit calculations. The difference to our case essentially is one of complexity: The dimension is four instead of two, there are 8 basic invariants satisfying 9 relations instead of Klein's three invariants with a single relation, and the singular locus is itself a complicated singular variety instead of an isolated point. The first construction that resolved the ADE-singularities in a uniform way was given by Kronheimer in [12] in terms of certain hyper-Kähler quotients. Later Ito and Nakamura [8] used G-Hilbert schemes to the same effect.

Recall that given a scheme X with an action of a finite group G the equivariant Hilbert scheme G-Hilb(X) is the moduli scheme of zero-dimensional equivariant subschemes $\xi \subset X$ such that $H^0(\xi, \mathcal{O}_\xi)$ is isomorphic to the regular representation of G. There is a canonical morphism G-Hilb$(X) \to X/G$ which is an isomorphism over the open subset that corresponds to regular orbits. For a linear action $G \subset \mathrm{SL}(V)$ it is known that G-Hilb(V) is smooth if $\dim(V) = 2$ or 3 and provides a crepant resolution of V/G (see [2]).

Thus $\rho : H := T\text{-Hilb}(S_1 \oplus S_2) \to Z = (S_1 \oplus S_2)/T$ is a natural candidate for a resolution. As the generic singularity of Z is a transversal A_2 singularity that arises from a $\mathbb{Z}/3$-action it is clear that ρ is a crepant resolution off the origin. However, it turns out that H has two irreducible components that are smooth and intersect transversely. One of them is the closure H^{orb} of the locus of regular orbits, it dominates the quotient Z. This orbit component also appears as 'dynamical component' or 'coherent component' in the literature. Any other component of H must be contained in the fibre $\rho^{-1}(0)$, though this is not true in general.

The two factors of the group $\mathbb{C}^* \times \mathbb{C}^*$ act on $S_1 \oplus S_2$ via dilations on the first and second summand, respectively, and the polynomial ring $\mathbb{C}[S_1 \oplus S_2]$ may accordingly be decomposed into irreducible $T \times \mathbb{C}^* \times \mathbb{C}^*$–representations, the first terms being

$$\begin{aligned}\mathbb{C}[S_1 \oplus S_2] \quad &= \quad L_0 \oplus (S_1 x \oplus S_2 y) \oplus R_0 x^2 \oplus (R_0 \oplus L_0) xy \oplus R_0 y^2 \\ &\quad\quad \oplus (S_1 \oplus S_2)(x^3 \oplus y^3) \oplus (S_0 \oplus S_1 \oplus S_2)(x^2 y \oplus xy^2) \oplus \ldots\end{aligned}$$

where x and y are formal symbols indicating the weight with respect to the $\mathbb{C}^* \times \mathbb{C}^*$ action. Using this decomposition one can see that H contains a further component isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$: if $I \in H$ is to be an ideal contained in the square of the maximal ideal generated by $S_1 x \oplus S_2 y$, of the respectively three copies of S_1 and S_2 of total weight 3 two have to be contained in I. The possible choices amount to picking a line in a three-dimensional space for each of S_1 and S_2. Of course, one still needs to check that every choice really leads to an admissable ideal.

As the map ρ is proper, each equivariant closed subset of H must contain fixed points for the $\mathbb{C}^* \times \mathbb{C}^*$-action. These correspond to T-equivariant bihomogeneous ideals $I \subset \mathbb{C}[S_1 \oplus S_2]$. Using the given decomposition of the coordinate ring it is not difficult to see that there are 13 such fixed points $I_i \in H$. The tangent space to H at I_i ist given by $\mathrm{Hom}_T(I_i, \mathbb{C}[S_1 \oplus S_2]/I_i)$. An explicit calculation shows that the dimension of the tangent space is 4 in seven points (necessarily smooth points of H) and is 5 in six other points. The calculation of the quadratic component of the analytic obstruction or Kuranishi map $\mathrm{Hom}_T(I_i, \mathbb{C}[S_1 \oplus S_2]/I_i) \to \mathrm{Ext}^1_T(I_i, \mathbb{C}[S_1 \oplus S_2]/I_i)$ yields in all cases a reducible quadric with two distinct factors. This suffices to conclude that there are no further components of H, that H^{orb} is smooth and that the two components H^{orb} and $\mathbb{P}^2 \times \mathbb{P}^2$ intersect transversely. By the universal property of the blow-up there is a commuting diagram

$$\begin{tikzcd}[column sep=small] & H^{\mathrm{orb}} \arrow[dl] \arrow[dr] & \\ Z_1'' \arrow[dr] & & Z_2'' \arrow[dl] \\ & Z & \end{tikzcd},$$

which conjecturely relates the two-resolutions by a Mukai-flop. In fact, we found the two resolutions first by contracting local models of H that are given as subschemes of a relative Grassmannian over Z. The calculations so far described are insufficient to formally prove that H^{orb} and $\mathbb{P}^2 \times \mathbb{P}^2$ intersect along the incidence variety and that this intersection is the exceptional locus for the two contractions. However, recall that equivariant Hilbert schemes can be seen as special cases of quiver varieties ([14], [11]) for the McKay quiver [13] associated to the given action. As the referee suggests one might try to obtain the diagram above and resolutions of Z in a single stroke by a variation of the stability condition in the construction of the quiver variety. As the McKay quiver for the action of T on S_0 is the Dynkin graph of type E_6, it is easy to deduce the graph underlying the McMay quiver for the action of T on $S_1 \oplus S_2$:

$$\begin{tikzcd}[column sep=small] & L_2 \arrow[r, dash] \arrow[dl, dash] & S_1 \arrow[dl, dash, shift left] \arrow[dl, dash, shift right] \arrow[dr, dash] & \\ S_0 \arrow[r, dash, shift left] \arrow[r, dash, shift right] \arrow[dr, dash] & R \arrow[dr, dash, shift left] \arrow[dr, dash, shift right] & & L_0 \arrow[dl, dash] \\ & L_1 \arrow[r, dash] & S_2 & \end{tikzcd}$$

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