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THE LINE BUNDLES ON THE MODULI OF PARABOLIC G-BUNDLES OVER CURVES AND THEIR SECTIONS

BY YVES LASZLO AND CHRISTOPH SORGER

RÉSUMÉ. – Soient X une courbe complexe, lisse, projective et connexe et G un groupe algébrique complexe, simple et simplement connexe. Nous calculons le groupe de Picard du champ des G -fibrés quasi-paraboliques sur X , décrivons explicitement ses générateurs pour G de type classique ou G_2 , puis identifions les espaces de sections globales correspondants avec les espaces de vacua de Tsuchiya, Ueno et Yamada. La méthode utilise le théorème d'uniformisation qui décrit ces champs comme doubles quotients de certains groupes algébriques de dimension infinie. Nous décrivons le fibré dualisant du champ des G -fibrés et montrons qu'il admet une unique racine carrée, que nous construisons explicitement. Si G n'est pas simplement connexe, la racine carrée dépend du choix d'une thêta-caractéristique. Ces résultats sur les champs permettent de retrouver le théorème de Drezet et Narasimhan (pour l'espace de modules grossier) et de montrer un énoncé analogue dans le cas $G = Sp_{2r}$. Nous montrons aussi que le module grossier des SO_r -fibrés n'est pas localement factoriel pour $r \geq 7$.

ABSTRACT. – Let X be a complex, smooth, complete and connected curve and G be a complex simple and simply connected algebraic group. We compute the Picard group of the stack of quasi-parabolic G -bundles over X , describe explicitly its generators for classical G and G_2 and then identify the corresponding spaces of global sections with the vacua spaces of Tsuchiya, Ueno and Yamada. The method uses the uniformization theorem which describes these stacks as double quotients of certain infinite dimensional algebraic groups. We describe also the dualizing bundle of the stack of G -bundles and show that it admits a unique square root, which we construct explicitly. If G is not simply connected, the square root depends on the choice of a theta-characteristic. These results about stacks allow to recover the Drezet-Narasimhan theorem (for the coarse moduli space) and to show an analogous statement when $G = Sp_{2r}$. We prove also that the coarse moduli spaces of semi-stable SO_r -bundles are not locally factorial for $r \geq 7$.

1. Introduction

(1.1) Fix a simple and simply connected algebraic group G over \mathbb{C} and a Borel subgroup $B \subset G$. Let X be a smooth, complete and connected curve over \mathbb{C} and p_1, \dots, p_n be distinct points of X , labeled by standard (i.e. containing B) parabolic subgroups P_1, \dots, P_n of G (we allow $n = 0$). Let $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$ be the moduli stack of quasi-parabolic G -bundles (cf. 8.3) of type $\underline{P} = (P_1, \dots, P_n)$ at $\underline{p} = (p_1, \dots, p_n)$ and denote by $X(P_i)$ the character group of P_i .

THEOREM. – *There is a line bundle \mathcal{L} on $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$ such that*

$$\gamma : \text{Pic}(\mathcal{M}_G^{par}(\underline{p}, \underline{P})) \xrightarrow{\sim} \mathbb{Z}\mathcal{L} \times \prod_{i=1}^n X(P_i).$$

If G is of type A_r or C_r ($r \geq 1$) then \mathcal{L} is the determinant line bundle (cf. 6.5); if G is of type B_r ($r \geq 3$), D_r ($r \geq 4$) or G_2 , then \mathcal{L} is the pfaffian line bundle (cf. 7.8) associated to the standard representation of G .

If G is of type E_6, E_7, E_8, F_4 , we believe that we can choose \mathcal{L} such that we have $\mathcal{L}^{\otimes d(G)} = \mathcal{D}_{\rho(G)}$, where respectively $d(G) = 6, 12, 60, 6$ and $\mathcal{D}_{\rho(G)}$ is the determinant line bundle (cf. 6.5) associated respectively to the fundamental representation $\rho(G) = \varpi_6, \varpi_7, \varpi_8, \varpi_4$ (cf. the discussion in 1.4).

(1.2) Suppose that the points \underline{p} are instead labeled by finite dimensional irreducible representations $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ of G and that an additional integer ℓ , the *level*, is fixed. The choice of a representation λ of G is equivalent to the choice of a standard parabolic subgroup $P \subset G$ and a dominant (with respect to B) character $\chi \in X(P)$ (cf. 8.1). Therefore, the labeling of the points \underline{p} by the representations $\underline{\lambda}$ defines the type \underline{P} of a quasi-parabolic G -bundle, that is the stack $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$ and, by the above theorem, a line bundle $\mathcal{L}(\ell, \underline{\chi})$ over $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$. The global sections of $\mathcal{L}(\ell, \underline{\chi})$ give a vector space, the space of *generalized parabolic G -theta-functions of level ℓ* , which is canonically associated to $(X, \underline{p}, \underline{\lambda})$. In mathematical physics, the rational conformal field theory of Tsuchiya, Ueno and Yamada [31] associates also to $(X, \underline{p}, \underline{\lambda}, \ell)$ a vector space: the space of *conformal blocks* $V_X(\underline{p}, \underline{\lambda}, \ell)$ (cf. [29] for an overview).

THEOREM. – Suppose that G is classical or G_2 . There is a canonical isomorphism

$$(1.2.1) \quad H^0(\mathcal{M}_G^{par}(\underline{p}, \underline{P}), \mathcal{L}(\ell, \underline{\chi})) \xrightarrow{\sim} V_X(\underline{p}, \underline{\lambda}, \ell).$$

In particular, $\dim H^0(\mathcal{M}_G^{par}(\underline{p}, \underline{P}), \mathcal{L}(\ell, \underline{\chi}))$ is given by the Verlinde formula.

For $n = 0$, this has been proved independently by Beauville and the first author [3] for $G = SL_r$ and by Faltings [10] and Kumar, Narasimhan and Ramanathan [16] for arbitrary simple and simply connected G . For arbitrary n and $G = SL_r$ this has been proved by Pauly [23] and will be proved in section 8 for arbitrary simple and simply connected G and arbitrary n , using (1.3) (and therefore [8]) and (5.1) below, following the lines of [3] and [23].

(1.3) The above results are proved via the *uniformization* theorem. Restrict for simplicity of the introduction to $n = 0$. Suppose $p \in X$ and denote $X^* = X - p$. Define $D = \text{Spec}(\hat{\mathcal{O}}_p)$, where $\hat{\mathcal{O}}_p$ is the formal completion of the local ring \mathcal{O}_p at p and $D^* = \text{Spec}(K_p)$ where K_p is the quotient field of $\hat{\mathcal{O}}_p$. Let LG (resp. L^+G , resp. L_XG) be the group of algebraic morphisms from D^* (resp. D , resp. X^*) to G (cf. 3.6 for the precise definition).

THEOREM. – The algebraic stack \mathcal{M}_G is canonically isomorphic to the double quotient stack $L_XG \backslash LG / L^+G$. Moreover, the projection

$$\mathcal{Q}_G := LG / L^+G \xrightarrow{\pi} \mathcal{M}_G$$

is locally trivial for the étale topology.

This is proved in [3] for $G = SL_r$. The extension to arbitrary semi-simple G has been made possible by Drinfeld and Simpson [DS] in response to a question by the first author.

They prove that if S is a \mathbb{C} -scheme and E a G -bundle over $X \times S$ then, locally for the étale topology on S , the restriction of E to $X^* \times S$ is trivial, which is essential for the proof. The above theorem is valid more generally for semi-simple, not necessarily simply connected G over an algebraically closed field k , after replacing “étale” by “fppf” if the characteristic of k divides the order of $\pi_1(G(\mathbb{C}))$.

(1.4) Consider the pullback morphism, deduced from π of 1.3,

$$\pi^* : \text{Pic}(\mathcal{M}_G) \longrightarrow \text{Pic}(\mathcal{Q}_G).$$

The Picard group of \mathcal{Q}_G is known ([20], [16]) to be canonically isomorphic to \mathbb{Z} , which reduces proving Theorem 1.1 to proving that π^* is an isomorphism. We will show that the injectivity of α will follow from the fact that $L_X G$ has no characters which in turn will follow from the fact that $L_X G$ is reduced and connected. Moreover, the surjectivity of α would follow from the simple connectedness of $L_X G$. Both topological properties, connectedness and simple connectedness of $L_X G$ are affirmed in [16] and we believe them to be true. Whereas we will prove the connectedness of $L_X G$, following an idea of V. Drinfeld, we do not see how to prove the simple connectedness of $L_X G$. The injectivity is enough to prove the first part of Theorem 1.1, but to identify the generator \mathcal{L} we should prove the surjectivity of α . For classical G and G_2 we do this by constructing in (7.8) a line bundle on \mathcal{M}_G , called the pfaffian line bundle, which will pull back to a generator of $\text{Pic}(\mathcal{Q}_G)$.

(1.5) The construction of the pfaffian line bundle (7.8) may be used to prove the following Proposition, valid more generally for semi-simple, not necessarily simply connected G over any algebraically closed field of characteristic $\neq 2$.

PROPOSITION. – *For every theta-characteristic κ on X , there is a canonical square-root \mathcal{P}_κ of the dualizing sheaf $\omega_{\mathcal{M}_G}$ of \mathcal{M}_G .*

(1.6) The last section will be devoted to Ramanathan's moduli spaces M_G of semi-stable G -bundles. We will show how some of the results for the stack \mathcal{M}_G will be true also for the moduli spaces M_G . In particular we will recover (and extend) the Drezet-Narasimhan theorem.

THEOREM. – *There is a canonical isomorphism $\text{Pic}(M_G) \xrightarrow{\sim} \mathbb{Z}L$. If G is of type A or C then L is the determinant bundle and moreover M_G is locally factorial in this case. If G is of type B_r ($r \geq 3$), D_r ($r \geq 4$) or G_2 then L or $L^{\otimes 2}$ is the determinant bundle.*

This theorem has also been proved, independently and with a different method, by Kumar and Narasimhan [17].

The Picard groups of \mathcal{M}_G and M_G for semi-simple, not necessarily simply connected G , and the question for which G exactly, M_G is locally factorial are studied in forthcoming joint work with Beauville. We show there for example that $\text{Pic}(M_{\text{Spin}_r})$ is generated by the determinant line bundle and in particular that M_{Spin_r} is not locally factorial for $r \geq 7$ by “lifting” to Spin_r , the proof we give here (9.5) for the analogous statement for M_{SO_r} .

We would like to thank A. Beauville and C. Simpson for useful discussions and V. Drinfeld for his suggestion in (5.1) and for pointing out an inaccuracy in an earlier version of this paper.

2. Some Lie theory

(2.1) Let \mathfrak{g} be a simple finite dimensional Lie algebra over \mathbb{C} . We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by Δ the associated root system. We have the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha})$. The Lie subalgebra $\mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{\alpha}$, isomorphic as a Lie algebra to \mathfrak{sl}_2 , will be denoted by $\mathfrak{sl}_2(\alpha)$. Moreover we choose a basis $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of Δ and we denote by Δ_+ the set of positive roots (with respect to Π). Put $\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_+} \mathfrak{g}_{\alpha})$. For each $\alpha \in \Delta_+$, we denote by H_{α} the coroot of α , i.e. the unique element of $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_{\alpha}) = 2$, and we denote by $X_{\alpha} \in \mathfrak{g}_{\alpha}$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ the elements such that $[H_{\alpha}, X_{\alpha}] = 2X_{\alpha}$ and $[H_{\alpha}, X_{-\alpha}] = -2X_{-\alpha}$. When α is one of the simple roots α_i , we write H_i, X_i, Y_i instead of $H_{\alpha_i}, X_{\alpha_i}, Y_{\alpha_i}$. Let (ϖ_i) be the basis of \mathfrak{h}^* dual to the basis (H_i) . Let P be the weight lattice and $P_+ \subset P$ be the set of dominant weights. Given a dominant weight λ , denote L_{λ} the associated simple \mathfrak{g} -module with highest weight λ and v_{λ} its highest weight vector. Finally (\cdot, \cdot) will be the Cartan-Killing form normalized such that for the highest root θ we have $(\theta, \theta) = 2$.

(2.2) Let $L\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ be the loop algebra of \mathfrak{g} . We will also consider its sub-Lie-algebras $L^+\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[z]]$, $L^{>0}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} z\mathbb{C}[[z]]$, and $L^{<0}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} z^{-1}\mathbb{C}[[z^{-1}]]$. There is a natural 2-cocycle

$$\psi_{\mathfrak{g}} : L\mathfrak{g} \times L\mathfrak{g} \longrightarrow \mathbb{C}$$

$$(X \otimes f, Y \otimes g) \mapsto (X, Y) \text{Res}(gdf)$$

defining a central extension $\widehat{L\mathfrak{g}}$ of \mathfrak{g} :

$$(2.2.1) \quad 0 \longrightarrow \mathbb{C} \longrightarrow \widehat{L\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0.$$

Let $\widehat{L^+\mathfrak{g}}$ be the extension of $L^+\mathfrak{g}$ obtained by restricting the above extension to $L^+\mathfrak{g}$. As the cocycle is trivial over $L^+\mathfrak{g}$ this extension splits.

Let ℓ be a positive integer. A representation of $\widehat{L\mathfrak{g}}$ is of level ℓ if the center c acts by multiplication by ℓ . Such a representation is called integrable if $X \otimes f$ acts locally nilpotent for all $X \otimes f \in \mathfrak{g}_{\alpha} \otimes_{\mathbb{C}} \mathbb{C}((z))$. The theory of affine Lie algebras [Kac] affirms that the irreducible integrable representations of level ℓ of $\widehat{L\mathfrak{g}}$ are classified (up to isomorphism) by the weights $P_{\ell} = \{\lambda \in P_+ / (\theta, \lambda) \leq \ell\}$. We denote by $\mathcal{H}_{\lambda}(\ell)$ the irreducible integrable representation of level ℓ and highest weight $\lambda \in P_{\ell}$. In the sequel we will use the following facts, which follow from the construction of $\mathcal{H}_{\lambda}(\ell)$:

$$(2.2.2) \quad L_{\lambda} = [\mathcal{H}_{\lambda}(\ell)]^{L^{>0}\mathfrak{g}} = \{v \in \mathcal{H}_{\lambda}(\ell) / L^{>0}\mathfrak{g}.v = 0\}$$

$$(2.2.3) \quad \mathcal{H}_{\lambda}(\ell) \text{ is generated by } L_{\lambda} \text{ over } L^{<0}\mathfrak{g} \text{ with only one relation:}$$

$$(X_{\theta} \otimes z^{-1})^{\ell - (\theta, \lambda) + 1}.v_{\lambda} = 0$$

If $\lambda = 0$, the corresponding representation, which we denote simply by $\mathcal{H}(\ell)$, is called the *basic representation of level ℓ* .

(2.3) Let $\rho : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$ be a representation of \mathfrak{g} . Then ρ induces a morphism of Lie algebras $L\mathfrak{g} \rightarrow L\mathfrak{sl}(V)$ and we can pull back the central extension (2.2.1):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & L\widehat{\mathfrak{sl}(V)} & \longrightarrow & L\mathfrak{sl}(V) & \longrightarrow & 0 \end{array}$$

The cocycle of the central extension $\widetilde{L\mathfrak{g}}$ is of the type $d_\rho \psi_\theta$. Define the *Dynkin index* of the representation ρ of \mathfrak{g} by the number d_ρ .

LEMMA 2.4. – Let $V = \sum_\lambda n_\lambda e^\lambda$ be the formal character of V . Then we have

$$d_\rho = \frac{1}{2} \sum_\lambda n_\lambda \lambda(H_\theta)^2$$

where θ is the highest root.

Proof. – By definition of the cocycle, we have $d_\rho = \text{Tr}(\rho(X_\theta)\rho(X_{-\theta}))$. Decompose the $\mathfrak{sl}_2(\theta)$ -module V as $\oplus_i V^{(d_i)}$, where $V^{(d_i)}$ is the standard irreducible \mathfrak{sl}_2 -module with highest weight d_i . We may realize $V^{(d_i)}$ as the vector space of homogeneous polynomials in 2 variables x and y of degree d_i . Then X_θ acts as $x\partial/\partial y$, and $X_{-\theta}$ as $y\partial/\partial x$. Using the basis $x^l y^{d_i-l}$, $l = 0, \dots, d_i$ of $V^{(d_i)}$, we see

$$d_\rho = \sum_i \sum_{k=0}^{d_i} k(d_i + 1 - k).$$

The formal character of the $\mathfrak{sl}_2(\theta)$ -module $V^{(d)}$ is $\sum_{k=0}^d e^{d\rho_\theta - k\alpha_\theta}$ where α_θ is the positive root of $\mathfrak{sl}_2(\theta)$ and $\rho_\theta = \frac{1}{2}\alpha_\theta$. Therefore we are reduced to prove the equality

$$\sum_{k=0}^d k(d+1-k) = \frac{1}{2} \sum_{k=0}^d ((d\rho_\theta - k\alpha_\theta)(H_\theta))^2 = \frac{1}{2} \sum_{k=0}^d (d-2k)^2$$

which is easy. ■

REMARK 2.5. – The Dynkin index ([6], Ch.1, §2) of a representation has been introduced to the theory of G -bundles over a curve by Faltings [10] and Kumar, Narasimhan, Ramanathan [16].

We are interested here in the *Dynkin index* of \mathfrak{g} , denoted $d_\mathfrak{g}$, which we define to be the greatest common divisor of the d_ρ where ρ runs over all representations $\rho : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$.

PROPOSITION 2.6. – The Dynkin index of \mathfrak{g} is as follows

Type of \mathfrak{g}	A_r	B_r ($r \geq 3$)	C_r	D_r ($r \geq 4$)	E_6	E_7	E_8	F_4	G_2
$d_\mathfrak{g}$	1	2	1	2	6	12	60	6	2
λ s.t. $d_\mathfrak{g} = d_{\rho(\lambda)}$	ϖ_1	ϖ_1	ϖ_1	ϖ_1	ϖ_6	ϖ_7	ϖ_8	ϖ_4	ϖ_1

Moreover, for any representation $\rho : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$, we have $d_\rho = 0 \bmod d_\mathfrak{g}$.

Proof. – Using the fact that $d_{V \otimes W} = r_W d_V + r_V d_W$ if V and W are two \mathfrak{g} -modules of rank r_V and r_W ([6], Ch.1, §2) proving the proposition reduces to prove it for the fundamental weights where the Dynkin numbers are known from Dynkin ([6], Table 5). We reproduce his numbers for the convenience of the reader (and correct some misprints in his table):

A_r	$d_{\varpi_i} = \binom{r-1}{i-1}$ for $i = 1, \dots, r$
$B_r (r \geq 3)$	$d_{\varpi_i} = 2 \binom{2r-1}{i-1}$ for $i = 1, \dots, r-1$ and $d_{\varpi_r} = 2^{(r-2)}$
C_r	$d_{\varpi_i} = \binom{2r-2}{i-1} - \binom{2r-2}{i-3}$ for $i = 1, \dots, r$
$D_r (r \geq 4)$	$d_{\varpi_i} = 2 \binom{2r-2}{i-1}$ for $i = 1, \dots, r-2$ and $d_{\varpi_{r-1}} = d_{\varpi_r} = 2^{(r-3)}$

For non classical \mathfrak{g} , we have

	d_{ϖ_1}	d_{ϖ_2}	d_{ϖ_3}	d_{ϖ_4}	d_{ϖ_5}	d_{ϖ_6}	d_{ϖ_7}	d_{ϖ_8}
G_2	2	8						
F_4	18	882	126	6				
E_6	6	24	150	1800	150	6		
E_7	36	360	4680	297000	17160	648	12	
E_8	1500	85500	5292000	8345660400	141605100	1778400	14700	60

3. The stack \mathcal{M}_G

In this section, k will be an algebraically closed field, G a reductive algebraic group over k and X a smooth, complete and connected curve over k .

(3.1) Let Z be a scheme over k . Recall that a *principal G -bundle* over Z (or just G -bundle for short), is a scheme $E \rightarrow Z$ equipped with a right action of G such that, locally in the flat topology, E is trivial, i.e. isomorphic to $G \times Z$ as a G -homogeneous space. In particular, E is affine, flat and smooth over Z . Moreover, the above conditions imply that E is even locally trivial for the étale topology.

If F is a quasi-projective scheme on which G acts on the left and E is a G -bundle, we can form $E(F) = E \times^G F$ the *associated bundle with fiber F* . It is the quotient of $E \times F$ under the action of G defined by $g.(e, f) = (e.g, g^{-1}f)$. Let $\rho : G \rightarrow G'$ be a morphism of algebraic groups. Then, as G acts on G' via ρ , we can form the *extension of the structure group* of a G -bundle E , that is the G' -bundle $E(G')$. Conversely, if F is a G' -bundle, a *reduction of structure group* F_G is a G -bundle E together with an isomorphism $F_G(G') \xrightarrow{\sim} F$. If ρ is faithful, such reductions are in one to one correspondence with sections of the associated bundle $F/G = F(G'/G)$.

(3.2) Let us collect some well known generalities on stacks for further reference. Let Aff/k be the flat affine site over k , that is the category of k -algebras equipped with the *fppf* topology. By *k -space* (resp. *k -group*) we understand a sheaf of sets (resp. groups)

over Aff/k . Any k -scheme can (and will) be considered as a k -space. The category of k -spaces is closed under direct limits. A k -space (k -group) will be called an *ind-scheme* (resp. *ind-group*) if it is direct limit of a directed system of schemes. Remark that an ind-group is not necessarily an inductive limit of algebraic groups. We will view k -stacks from the pseudo-functorial point of view, i.e. a k -stack \mathfrak{X} will associate to every k -algebra R a groupoid $\mathfrak{X}(R)$ and to every morphism of k -algebras $u : R \rightarrow R'$ a functor $u^* : \mathfrak{X}(R') \rightarrow \mathfrak{X}(R)$ together with isomorphisms of functors $(u \circ v)^* \simeq v^* \circ u^*$ satisfying the usual cocycle condition. The required topological properties are that for every $x, y \in \text{ob} \mathfrak{X}(R)$ the presheaf $\underline{\text{Isom}}(x, y)$ is a sheaf and that all descent data are effective ([18], 2.1). Any k -space may be seen as a k -stack, by considering a set as a groupoid (with the identity as the only morphism). Conversely, any k -stack \mathfrak{X} such that $\mathfrak{X}(R)$ is a discrete groupoid (i.e. has only the identity as automorphisms) for all k -algebras R , is a k -space. A morphism $F : \mathfrak{X} \rightarrow \mathfrak{Y}$ will associate, for every k -algebra R , a functor $\mathfrak{X}(R) \rightarrow \mathfrak{Y}(R)$ satisfying the obvious compatibility conditions. Let $S = \text{Spec}(R)$ and consider a morphism $\eta : S \rightarrow \mathfrak{Y}$, that is an object η of $\mathfrak{Y}(S)$. The fiber \mathfrak{X}_η is a stack over S . The morphism F is *representable* if \mathfrak{X}_η is representable as an algebraic space for all $S = \text{Spec}(R)$. A stack \mathfrak{X} is *algebraic* if the diagonal morphism $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is representable, separated and quasi-compact and if there is an algebraic space X and a representable, smooth, surjective morphism of stacks $P : X \rightarrow \mathfrak{X}$.

Suppose Z is a k -space and that the k -group Γ acts on Z . Then the quotient stack $[Z/\Gamma]$ is defined as follows. Let R be a k -algebra. The objects of $[Z/\Gamma](R)$ are pairs (E, α) where E is a Γ -bundle (the definition in (3.1) makes of course sense for k -groups) over $\text{Spec}(R)$ and $\alpha : E \rightarrow Z$ is G -equivariant. The arrows are defined in the obvious way and so are the functors $[Z/\Gamma](R') \rightarrow [Z/\Gamma](R)$.

(3.3) We denote by \mathcal{M}_G the stack of G -bundles over X . It is defined as follows. For any k -algebra R denote X_R the scheme $X \times_k \text{Spec}(R)$. Then objects of $\mathcal{M}_G(R)$ are G -bundles over X_R and morphisms of $\mathcal{M}_G(R)$ are isomorphisms of G -bundles.

PROPOSITION 3.4. – *The stack \mathcal{M}_G is algebraic and smooth. Moreover we have $\dim \mathcal{M}_G = (g - 1) \dim G$.*

The above proposition is well known. That \mathcal{M}_G is algebraic for $G = GL_r$ is the content of theorem 4.14.2.1 of [LMB]. For general (reductive) G consider an embedding $G \subset GL_r$. Using the fact that if $\mathfrak{X} \rightarrow \mathfrak{Y}$ is a representable morphism of stacks then \mathfrak{X} is algebraic if \mathfrak{Y} is, it is then enough to show that for an embedding $\rho : G \hookrightarrow G'$ the morphism of stacks $\mathcal{M}_G \rightarrow \mathcal{M}_{G'}$ given by extension of the structure group is representable. But this follows from Grothendieck's theory of Hilbert schemes as reductions of the structure group of a G' -bundle F to G correspond to sections of F/G (cf. 3.1).

(3.5) Choose a closed point p on X and set $X^* = X - p$. Let $\widehat{\mathcal{O}}_p$ be the completion of the local ring of X at p , and K_p its field of fractions. Set $D = \text{Spec}(\widehat{\mathcal{O}}_p)$ and $D^* = \text{Spec}(K_p)$. We choose a local coordinate z at p and identify $\widehat{\mathcal{O}}_p$ with $k[[z]]$ and K_p with $k((z))$. Let R be a k -algebra. Define $X_R = X \times_k \text{Spec}(R)$, $X_R^* = X^* \times_k \text{Spec}(R)$, $D_R = \text{Spec}(R[[z]])$

and $D_R^* = \text{Spec}(R((z)))$. Then we have the cartesian diagram

$$\begin{array}{ccc} D_R^* & \longrightarrow & D_R \\ \downarrow & & \downarrow \\ X_R^* & \longrightarrow & X_R \end{array}$$

We denote by A_{X_R} the k -algebra $\Gamma(X_R^*, \mathcal{O}_{X_R^*})$.

(3.6) We denote by LG the *loop group* of G that is the k -group defined by $R \mapsto G(R((z)))$, where R is any k -algebra. The group of *positive loops*, that is the k -group $R \mapsto G(R[[z]])$ will be denoted by L^+G and the group of *negative loops*, that is the k -group $R \mapsto G(R[z^{-1}])$ will be denoted by L^-G . The group of *loops coming from X^** , i.e. the k -group defined by $R \mapsto G(A_{X_R})$, will be denoted by $L_X G$. Finally, we will also use the k -group $L^{<0}G$ defined by $R \mapsto G(z^{-1}R[z^{-1}])$. The Lie algebra of LG is $L\mathfrak{g}$, as the kernel of the homomorphism $LG(R[\varepsilon]) \rightarrow LG(R)$ is $L\mathfrak{g}(R) = \mathfrak{g} \otimes_{\mathbb{C}} R((z))$. For the same reason we have $\text{Lie}(L^+G) = L^+\mathfrak{g}$ and $\text{Lie}(L_X G) = L_X \mathfrak{g}$. We denote \mathcal{Q}_G the quotient k -space LG/L^+G which we discuss in more detail in Section 4.

(3.7) Consider the functor T_G which associates to a k -algebra R the set $T_G(R)$ of isomorphism classes of triples (E, ρ, σ) where E is a G -bundle over X_R , $\rho : G \times X_R^* \rightarrow E|_{X_R^*}$ a trivialization of E over X_R^* and $\sigma : G \times D_R \rightarrow E|_{D_R}$ a trivialization of E over D_R .

PROPOSITION 3.8. – *The k -group LG represents the functor T_G .*

Proof. – Let (E, ρ, σ) be an element of $T_G(R)$. Pulling back the trivializations ρ and σ to D_R^* provides two trivializations ρ^* and σ^* of the pull back of E over D_R^* : these trivializations differ by an element $\gamma = \rho^{*-1} \circ \sigma^*$ of $G(R((z)))$. Conversely, let us start from an element γ of $G(R((z)))$. This element defines an isomorphism of the pullbacks over D_R^* of the trivial G -bundle over X_R^* and the trivial G -bundle over D_R . These two G -bundles glue together to a G -bundle E in a functorial way by [BL2] (in fact [BL2] is written for SL_r but the extension to G is straightforward). These constructions are inverse to each other by construction. ■

(3.9) Consider the functor P_G which associates to a k -algebra R the set $P_G(R)$ of isomorphism classes of pairs (E, ρ) , where E is a G -bundle over X_R and ρ a trivialization of E over X_R^* .

PROPOSITION 3.10. – *The k -space \mathcal{Q}_G represents the functor P_G .*

Proof. – Let R be a k -algebra and q an element of $\mathcal{Q}_G(R)$. By definition there exists a faithfully flat homomorphism $R \rightarrow R'$ and an element γ of $G(R'((z)))$ such that the image of q in $\mathcal{Q}_G(R')$ is the class of γ . To γ corresponds by Proposition 3.8 a triple (E', ρ', σ') over $X_{R'}$. Let $R'' = R' \otimes_R R'$, and let (E''_1, ρ''_1) , (E''_2, ρ''_2) denote the pull-backs of (E', ρ') by the two projections of $X_{R''}$ onto $X_{R'}$. Since the two images of γ in $G(R''((z)))$ differ by an element of $G(R''[[z]])$, these pairs are isomorphic. So the isomorphism $\rho''_2 \rho''_1{}^{-1}$ over $X_{R''}^*$ extends to an isomorphism $u : E''_1 \rightarrow E''_2$ over $X_{R''}$, satisfying the usual cocycle condition (it is enough to check this over X^* , where it is obvious). Therefore (E', ρ') descends to a pair (E, ρ) on X_R as in the statement of the proposition. Conversely, given a pair (E, ρ) as above over X_R , we can find a faithfully flat homomorphism $R \rightarrow R'$ and a trivialization σ' of the pull back of E over $D_{R'}$ (after base change, we may assume

that the central fiber of the restriction of E to D_R has a section then use smoothness to extend this section to D_R . By Proposition 3.8 we get an element γ' of $G(R'((z)))$ such that the two images of γ' in $G(R''((z)))$ (with $R'' = R' \otimes_R R'$) differ by an element of $G(R''[[z]])$; this gives an element of $\mathcal{Q}_G(R)$. These constructions are inverse to each other by construction. ■

We will make use of the following theorem of Drinfeld and Simpson:

THEOREM. – [8] *Suppose G is semi-simple. Let R be a k -algebra and E be a G -bundle over X_R . Then the restriction of E to X_R^* is trivial, locally for the fppf topology over $\text{Spec}(R)$. If $\text{char}(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$, then this is even true locally for the étale topology over $\text{Spec}(R)$.*

3.12. *Proof of Theorem 1.3.* – The universal G -bundle over $X \times \mathcal{Q}_G$ (Proposition 3.10), gives rise to a map $\pi : \mathcal{Q}_G \rightarrow \mathcal{M}_G$. This map is $L_X G$ -invariant, hence induces a morphism of stacks $\bar{\pi} : L_X G \backslash \mathcal{Q}_G \rightarrow \mathcal{M}_G$. On the other hand we can define a map $\mathcal{M}_G \rightarrow L_X G \backslash \mathcal{Q}_G$ as follows. Let R be a k -algebra, E a G -bundle over X_R . For any R -algebra R' , let $T(R')$ be the set of trivializations ρ of $E_{R'}$ over $X_{R'}^*$. This defines a R -space T on which the group $L_X G$ acts. By Theorem 3.11, it is a $L_X G$ -bundle. To any element of $T(R')$ corresponds a pair $(E_{R'}, \rho)$, hence by Proposition 3.10 an element of $\mathcal{Q}_G(R')$. In this way we associate functorially to an object E of $\mathcal{M}_G(R)$ a $L_X G$ -equivariant map $\alpha : T \rightarrow \mathcal{Q}_G$. This defines a morphism of stacks $\mathcal{M}_G \rightarrow L_X G \backslash \mathcal{Q}_G$ which is the inverse of $\bar{\pi}$. The second assertion means that for any scheme S over k (resp. over k such that $\text{char}(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$) and any morphism $f : T \rightarrow \mathcal{M}_G$, the pull back to S of the fibration π is fppf (resp. étale) locally trivial, i.e. admits local sections (for the fppf (resp. étale) topology). Now f corresponds to a G -bundle E over $X \times S$. Let $s \in S$. Again by Theorem 3.11, we can find an fppf (resp. étale) neighborhood U of s in S and a trivialization ρ of $E|_{X^* \times U}$. The pair (E, ρ) defines a morphism $g : U \rightarrow \mathcal{Q}_G$ (Proposition 3.10) such that $\pi \circ g = f$, that is a section over U of the pull back of the fibration π . ■

4. The infinite grassmannian \mathcal{Q}_G

Throughout this section we suppose G semi-simple and simply connected over \mathbb{C} . We will gather together some results on \mathcal{Q}_G which we will use later (cf. Faltings [10]).

(4.1) Choose a faithful representation $G \subset SL_r$. For $N \geq 0$, we denote by $LG(N)(R)$ the set of matrices $A(z)$ in $G(R((z))) \subset SL_r(R((z)))$ such that for both $A(z)$ and $A(z)^{-1}$, the coefficients have a pole of order $\leq N$. This defines a subfunctor $LG(N)$ of LG which is obviously representable by an (infinite dimensional) affine \mathbb{C} -scheme. In particular, the \mathbb{C} -group $L^+G = LG(0)$ is an affine group scheme. The \mathbb{C} -group LG is an ind-group, direct limit of the sequence of the schemes $(LG(N))_{N \geq 0}$. This ind-structure does not depend on the embedding $G \subset SL_r$. The \mathbb{C} -group $L_X G$ has the structure of an ind-group induced by the one of LG .

(4.2) Let \mathcal{H} be an infinite dimensional vector space over \mathbb{C} . We define the \mathbb{C} -space $\text{End}(\mathcal{H})$ by $R \mapsto \text{End}(\mathcal{H} \otimes_{\mathbb{C}} R)$, the \mathbb{C} -group $GL(\mathcal{H})$ as the group of its units and

$PGL(\mathcal{H})$ by $GL(\mathcal{H})/G_m$. The \mathbb{C} -group LG acts on $L\mathfrak{g}$ by the adjoint action. We define the adjoint action of LG on $\widehat{L\mathfrak{g}}$ as follows:

$$\mathrm{Ad}(\gamma).(\alpha', s) = (\mathrm{Ad}(\gamma).\alpha', s + \mathrm{Res}_{z=0}(\gamma^{-1} \frac{d}{dz} \gamma, \alpha'))$$

where $\gamma \in LG(R)$, $\alpha = (\alpha', s) \in \widehat{L\mathfrak{g}}(R)$ and $(,)$ is the $R((z))$ -bilinear extension of the Cartan-Killing form normalized as in (2.1). Consider an integral highest weight representation $\bar{\pi} : \widehat{L\mathfrak{g}} \rightarrow \mathrm{End}(\mathcal{H})$. The basic result we will use in the sequel is the following:

PROPOSITION 4.3. – (Faltings) *Let R be a \mathbb{C} -algebra, $\gamma \in LG(R)$. Locally over $\mathrm{Spec}(R)$, there is an automorphism u of $\mathcal{H}_R = \mathcal{H} \otimes_{\mathbb{C}} R$, unique up to R^* , such that*

$$(4.3.1) \quad u\bar{\pi}_R(\alpha)u^{-1} = \bar{\pi}_R(\mathrm{Ad}(\gamma).\alpha)$$

for any $\alpha \in \widehat{L\mathfrak{g}}(R)$.

(4.4) An immediate corollary of the above proposition is that the representation $\bar{\pi}$ may be “integrated” to a (unique) algebraic projective representation of LG , i.e. that there is a morphism of \mathbb{C} -groups $\pi : LG \rightarrow PGL(\mathcal{H})$ whose derivate coincides with $\bar{\pi}$ up to homothety. Indeed, thanks to the unicity property the automorphisms u associated locally to γ glue together to define an element $\pi(\gamma) \in PGL(\mathcal{H})(R)$ and still because of the unicity property, π defines a morphism of \mathbb{C} -groups. The assertion on the derivative is consequence of (4.3.1). The above proposition (and its corollary) are proved in ([3], App. A) in the case SL_r ; its generalization to G is straightforward.

(4.5) The quotient \mathbb{C} -space $\mathcal{Q}_G := LG/L^+G$ is an ind-scheme, direct limit of a sequence of projective varieties $\mathcal{Q}_G(N)$. This is shown for $G = SL_r$ in [3] Corollary 2.4 (take $\mathcal{Q}_{SL_r}(N) = LSL_r(N)/L^+SL_r$) and in the general case there is an analogue lattice description. Recall that an ind-scheme is called *reduced* (resp. *irreducible*, *integral*) if it is a direct limit of an increasing sequence of reduced (resp. irreducible, integral) schemes. By Lemma 6.3 of [3] an ind-scheme is integral if and only if it is irreducible and reduced.

PROPOSITION 4.6. – *The ind-scheme \mathcal{Q}_G is integral.*

Proof. – First of all, remark that the multiplication map $\mu : L^{<0}G \times L^+G \longrightarrow LG$ is an open immersion (argue as in [BL1] Proposition 1.11 and use that if $Y \rightarrow S$ is proper with a section σ and $G \subset SL_r$ is reductive, then any G -bundle E trivial along σ and such that $E(SL_r)$ is trivial is itself trivial). Then, according to Faltings (see [BL1] for the case SL_r), the ind-group L^-G is integral. This may be seen by looking at $(L^-G)_{red}$ and using Shavarevich’s theorem that a closed immersion of irreducible ind-affine groups which is an isomorphism on Lie algebras, is an isomorphism [Sh]. Note that irreducibility is due to the fact that any element can be deformed to a constant in G ; that $\mathrm{Lie}(L^-G) \rightarrow \mathrm{Lie}(L^-G)_{red}$ is an isomorphism can be seen by using the fact that $\mathrm{Lie}(G)$ is generated by nilpotent elements. It follows that $L^{<0}G$, which is a semidirect product of G and L^-G , is integral. Finally as G is simply connected, we see that the ind-scheme \mathcal{Q}_G is integral: as by the above $L^{<0}G \rightarrow \mathcal{Q}_G$ is an open immersion, it is enough to show that \mathcal{Q}_G is irreducible. Using that connected ind-groups are irreducible ([Sh], Proposition 3) and the quotient

morphism $LG \rightarrow \mathcal{Q}_G$ we reduce to prove the connectedness of LG which follows from the simple connectedness of G . ■

The quotient LG/L^+G has also been constructed by Kumar [15] and Mathieu [20] as inductive limit of reduced projective Schubert varieties.

PROPOSITION 4.7. – *The ind-structure on \mathcal{Q}_G coincides with the one of Kumar and Mathieu.*

Proof. – Both Kumar and Mathieu define the structure of ind-variety on LG/L^+G using representation theory of Kac-Moody algebras; for instance Kumar, following Slodowy [Sl], considers the basic representation $\mathcal{H}(\ell)$ for a fixed ℓ , and a highest weight vector v_ℓ . The subgroup L^+G is the stabilizer of the line kv_ℓ in $\mathbb{P}(\mathcal{H}(\ell))$, so the map $g \mapsto gv_\ell$ induces an injection $i_\ell : LG/L^+G \hookrightarrow \mathbb{P}(\mathcal{H}(\ell))$. Let U be the subgroup of L^+G consisting of elements $A(z)$ such that $A(0)$ is in the unipotent part of a fixed Borel subgroup $B \subset G$; to each element w of the Weyl group is associated a “Schubert variety” X_w which is a finite union of orbits of U . It turns out that the image under i_ℓ of X_w is actually contained in some finite-dimensional projective subspace \mathbb{P}_w of $\mathbb{P}(\mathcal{H}(\ell))$, and is Zariski closed in \mathbb{P}_w . This defines on X_w a structure of reduced projective variety, and a structure of ind-variety on $LG/L^+G = \varinjlim X_w$.

By (4.4), the irreducible integrable representation $\mathcal{H}(\ell)$ of $\widehat{L\mathfrak{g}}$ can be integrated to a projective representation of LG . It follows that the above map i_ℓ is a morphism of ind-schemes of \mathcal{Q}_G into $\mathbb{P}(\mathcal{H}(\ell))$ (which is the direct limit of its finite-dimensional subspaces). But i_ℓ is even an *embedding*. It is injective by what we said above; let us check that it induces an injective map on the tangent spaces. Since it is equivariant under the action of LG it is enough to prove this at the origin ω of \mathcal{Q}_G . Then it follows from the fact that the annihilator of v_ℓ in the Lie algebra $L\mathfrak{g}$ is $L^+\mathfrak{g}$ (2.2.2). Therefore the restriction of i_ℓ to each of the subvarieties $\mathcal{Q}_G(N)$ is proper, injective, and injective on the tangent spaces, hence is an embedding (in some finite-dimensional projective subspace of $\mathbb{P}(\mathcal{H}(\ell))$). Each X_w is contained in some $\mathcal{Q}_G(N)$, and therefore is a closed subvariety of $\mathcal{Q}_G(N)_{\text{red}}$. Each orbit of U is contained in some X_w ; since the X_w ’s define an ind-structure, each $\mathcal{Q}_G(N)$ is contained in some X_w , so that $\mathcal{Q}_G(N)_{\text{red}}$ is a subvariety of X_w . Since \mathcal{Q}_G is the direct limit of the $\mathcal{Q}_G(N)_{\text{red}}$, the two ind-structures coincide. ■

(4.8) Consider the basic representation $\mathcal{H}(1)$ of $\widehat{L\mathfrak{g}}$ and the central extension

$$(4.8.1) \quad 1 \longrightarrow G_m \longrightarrow GL(\mathcal{H}(1)) \longrightarrow PGL(\mathcal{H}(1)) \longrightarrow 1.$$

As $\mathcal{H}(1)$ may be integrated to a projective representation of LG by (4.4), pulling back (4.8.1) to LG , defines a central extension to which we refer as the *canonical* central extension of LG :

$$(4.8.2) \quad 1 \longrightarrow G_m \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1$$

An easy calculation shows $\text{Lie}(\widehat{LG}) = \widehat{L\mathfrak{g}}$; in particular \widehat{LG} is non trivial.

LEMMA 4.9. – *The extension (4.8.2) splits canonically over L^+G .*

It suffices to show that the representation $\bar{\pi} : L^+\mathfrak{g} \rightarrow \text{End}(\mathcal{H})$ integrates to an algebraic representation $\pi : L^+\mathfrak{g} \rightarrow GL(\mathcal{H}(1))$, which follows from the fact that in the case

$\gamma \in L^+G(R)$ we can *normalize* the automorphism u of Proposition 4.3. Indeed, as $L_0 = [\mathcal{H}(1)]^{L^+\mathfrak{g}}$ by (2.2.2), it follows from (4.3.1) that u maps $(L_0)_R$ to $(L_0)_R$. Now $(L_0)_R$ is a free R -module of rank one, hence we may choose u (in a unique way) such that it induces the identity on $(L_0)_R$. ■

Let $G_m \times L^+G \xrightarrow{\chi_0} G_m$ be the character defined by the first projection. We define \mathcal{L}_χ as the line bundle on the homogeneous space $\mathcal{Q}_G = \widehat{L}G/\widehat{L}^+G$ associated to $\chi = \chi_0^{-1}$.

(4.10) The Picard group of \mathcal{Q}_G is known to be infinite cyclic ([16], [20]) and may be described as follows. Consider the morphism of \mathbb{C} -groups $\varphi : SL_2 \rightarrow LSL_2$ defined by (for R a \mathbb{C} -algebra)

$$SL_2(R) \xrightarrow{\phi} SL_2(R((z)))$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix}$$

and moreover the morphism of \mathbb{C} -groups $\psi : LSL_2 \rightarrow LG$ deduced from the map $SL_2 \rightarrow G$ associated to the highest root θ . Let $\varphi = \psi \circ \phi : SL_2 \rightarrow LG$. The Borel subgroup $B_2 \subset SL_2$ of upper triangular matrices maps to L^+G by construction, hence we get a morphism $\overline{\varphi} : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathcal{Q}_G$. An easy calculation shows that the derivative $\text{Lie}(\varphi)$ maps the standard \mathfrak{sl}_2 -triplet $\{e, f, h\} = \{X_\theta, X_{-\theta}, H_\theta\}$ to the \mathfrak{sl}_2 -triplet $\{X_{-\theta} \otimes z, X_\theta \otimes z^{-1}, -H_\theta\}$ of LG . It follows from the description of $\text{Pic}(\mathcal{Q}_G)$ of [16], that the pullback defines an isomorphism:

$$\overline{\varphi}^* : \text{Pic}(\mathcal{Q}_G) \xrightarrow{\sim} \text{Pic}(\mathbb{P}_{\mathbb{C}}^1)$$

LEMMA 4.11. – We have $\overline{\varphi}^*(\mathcal{L}_\chi) = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(1)$, i.e. $\text{Pic}(\mathcal{Q}_G) = \mathbb{Z}\mathcal{L}_\chi$.

The restriction of (4.8.2) to SL_2 splits, hence φ lifts to a morphism $\tilde{\varphi} : SL_2 \rightarrow \widehat{LG}$ and all we have to do is to calculate the character of $B_2 \rightarrow \widehat{L}^+G \xrightarrow{\chi} G_m$. In order to do this it is enough to calculate the character of B_2 on the SL_2 -module generated by v_0 . By (2.2.3) this is the standard representation, so we are done. ■

In the following we denote, in view of the above, \mathcal{L}_χ by $\mathcal{O}_{\mathcal{Q}_G}(1)$.

5. The ind-group L_XG

Throughout this section, G is semi-simple and simply connected over \mathbb{C} and X is a smooth, complete and connected curve over \mathbb{C} .

PROPOSITION 5.1. – The ind-group L_XG is integral.

COROLLARY 5.2. – Every character $\chi : L_XG \rightarrow G_m$ is trivial.

Proof. – The differential of χ , considered as a function on L_XG , is everywhere vanishing. Indeed, since χ is a group morphism, this means that the deduced Lie algebra morphism $\mathfrak{g} \otimes A_X \rightarrow k$ is zero (recall $A_X = \Gamma(X^*, \mathcal{O}_{X^*})$). But as the derived algebra $[\mathfrak{g} \otimes A_X, \mathfrak{g} \otimes A_X]$ is $[\mathfrak{g}, \mathfrak{g}] \otimes A_X$ and therefore equal to $\mathfrak{g} \otimes A_X$ (as \mathfrak{g} is simple), any Lie algebra morphism $\mathfrak{g} \otimes A_X \rightarrow k$ is trivial. As L_XG is integral we can write L_XG as the direct limit of integral

varieties V_n . The restriction of χ to V_n has again zero derivative and is therefore constant. For large n , the varieties V_n contain 1. This implies $\chi|_{V_n} = 1$ and we are done. ■

Proof of the proposition. – To see that the ind-group $L_X G$ is reduced, consider the étale trivial morphism $\bar{\pi} : \mathcal{Q} \rightarrow \mathcal{M}_G$. Locally for the étale topology, $\bar{\pi}$ is a product $\Omega \times L_X G$ (where Ω is an étale neighborhood of \mathcal{M}_G). Then use that \mathcal{Q} is reduced (Section 4). To prove irreducibility, as connected ind-groups are irreducible by Proposition 3 of [Sh], it is enough to show that $L_X G$ is connected.

The idea how to prove that $L_X G$ is connected is due to V. Drinfeld: consider distinct points p_1, \dots, p_i of X which are all distinct from p . Define $X_i^* = X - \{p, p_1, \dots, p_i\}$ and, for every \mathbb{C} -algebra R , define $X_{i,R}^* = X_i^* \times_k \text{Spec}(R)$. Denote by $A_{X_{i,R}}$ the \mathbb{C} -algebra $\Gamma(X_{i,R}^*, \mathcal{O}_{X_{i,R}^*})$ and by $L_X^i G$ the \mathbb{C} -group $R \mapsto G(A_{X_{i,R}})$. As $L_X G$, the \mathbb{C} -group $L_X^i G$ is an ind-group. The natural inclusion $A_{X_{i,R}} \subset A_{X_{i+1,R}}$ defines a closed immersion $f : L_X^i G \rightarrow L_X^{i+1} G$.

LEMMA 5.3. – *The closed immersion $L_X^i G \rightarrow L_X^{i+1} G$ defines a bijection*

$$\pi_0(L_X^i G) \xrightarrow{\sim} \pi_0(L_X^{i+1} G).$$

Proof. – Consider the morphism $L_X^{i+1} G \rightarrow LG$ defined by the developpement in Laurent series at p_{i+1} . We get a morphism $\phi_{i+1} : L_X^{i+1} G \rightarrow \mathcal{Q}_G^{p_{i+1}}$, where we denote $\mathcal{Q}_G^{p_{i+1}} = LG/L^+ G$. (of course $\mathcal{Q}_G = \mathcal{Q}_G^{p_{i+1}}$ but we emphasize here that we will consider the point p_{i+1} and not p .)

CLAIM. – The morphism $\phi_{i+1} : L_X^{i+1} G \rightarrow \mathcal{Q}_G^{p_{i+1}}$ induces an isomorphism on the level of stacks $\bar{\phi}_{i+1} : L_X^{i+1} G / L_X^i G \simeq \mathcal{Q}_G^{p_{i+1}}$ and is locally trivial for the étale topology.

The lemma reduces to the claim. Indeed, as G is semi-simple and simply connected, we have $\pi_i([\mathcal{Q}_G^{p_{i+1}}]^{an}) = 1$ for $i = 0, 1$ (by Section 4 and Kumar and Mathieu) and the exact homotopy sequence associated of the (Serre)-fibration ϕ_{i+1} shows that $\pi_0([L_X^i G]^{an}) \xrightarrow{\sim} \pi_0([L_X^{i+1} G]^{an})$ ($[]^{an}$ means we consider the usual topology). From the bijection $\pi_0(L_X^i G(N)) \xrightarrow{\sim} \pi_0([L_X^i G(N)]^{an})$ and Proposition 2 of [Sh] it follows then that $\pi_0(L_X^i G) \rightarrow \pi_0([L_X^i G]^{an})$ is bijective.

Proof of the claim. – Clearly $\phi_{i+1} : L_X^{i+1} G \rightarrow \mathcal{Q}_G^{p_{i+1}}$ is $L_X^i G$ invariant, hence defines a map $\bar{\phi}_{i+1} : L_X^{i+1} G / L_X^i G \rightarrow \mathcal{Q}_G^{p_{i+1}}$. Define a morphism $\mathcal{Q}_G^{p_{i+1}} \rightarrow L_X^{i+1} G / L_X^i G$ as follows. Let R be a \mathbb{C} -algebra. By Proposition 3.10 to an element of $\mathcal{Q}_G^{p_{i+1}}(R)$ corresponds a G -bundle $E \rightarrow X_R$ together with a trivialization $\tau_{p_{i+1}} : G \times X_{p_{i+1},R}^* \rightarrow E|_{X_{p_{i+1},R}^*}$. Here by $X_{p_{i+1},R}^*$ we denote $(X - \{p_{i+1}\}) \times_k \text{Spec}(R)$. For any R -algebra R' , denote $T(R')$ the set of trivializations τ_i of $E_{R'}$ over $X_{i,R'}^*$. This defines a R -space T on which $L_X^i G$ acts. By Theorem 3.11 it is a $L_X^i G$ -bundle. For any $\tau_i \in T(R')$ the composite $\tau_i^{-1} \circ \tau_{p_{i+1}}$ defines a morphism $X_{i+1,R}^* \rightarrow G$ hence an element of $L_X^{i+1} G(R)$. In this way we associate functorially to an object $(E, \tau_{p_{i+1}})$ of $\mathcal{Q}_G^{p_{i+1}}(R)$ a $L_X^i G$ -invariant map $\alpha : T \rightarrow L_X^{i+1} G$, which defines the inverse of $\bar{\phi}$. The assertion concerning the local triviality is proved as in Theorem 1.3. ■

Let us show that every element $g \in L_X G(\mathbb{C})$ is in the connected component of the unit of $L_X G(\mathbb{C})$. Let K be the field of rational functions on X . Using the fact (cf. [30]) that

$G(K)$ is generated by the standard unipotent subgroups $U_\alpha(K)$, $\alpha \in \Delta$, we may suppose that g is of the form $\prod_{j \in J} \exp(f_j n_j)$ where the n_j are nilpotent elements of \mathfrak{g} and $f_j \in K$. Let $\{p_1, \dots, p_i\}$ be the poles of the functions f_j , $j \in J$. The morphism

$$\begin{aligned} \mathbb{A}^1 &\longrightarrow L_X^i G \\ t &\longmapsto \prod_{j \in J} \exp(t f_j n_j) \end{aligned}$$

is a path from g to 1 in $L_X^i G$. By Lemma 5.3, the morphism $\pi_0(L_X G) \rightarrow \pi_0(L_X^i G)$ is bijective which proves that g and 1 are indeed in the same connected component of $L_X G$. ■

6. The Picard group of \mathcal{M}_G

Throughout this section, G is simple and simply connected over \mathbb{C} and X is a smooth, complete and connected curve over \mathbb{C} .

Let $\text{Pic}_{L_X G}(\mathcal{Q}_G)$ be the group of $L_X G$ -linearized line bundles on \mathcal{Q}_G . Recall that a $L_X G$ -linearization of the line bundle \mathcal{L} on \mathcal{Q}_G is an isomorphism $m^* \mathcal{L} \xrightarrow{\sim} pr_2^* \mathcal{L}$, where $m : L_X G \times \mathcal{Q}_G \rightarrow \mathcal{Q}_G$ is the action of $L_X G$ on \mathcal{Q}_G , satisfying the usual cocycle condition.

Consider the projection $\pi : \mathcal{Q}_G \rightarrow \mathcal{M}_G$ of Theorem 1.3. Let \mathcal{L} be a line bundle on \mathcal{M}_G . As π^* induces an isomorphism between the sections of \mathcal{L} and $L_X G$ -invariant sections of $\pi^* \mathcal{L}$ ([3], Lemma 7.2), we have

PROPOSITION 6.1. – *The projection $\pi : \mathcal{Q}_G \rightarrow \mathcal{M}_G$ induces an injection*

$$\pi^* : \text{Pic}(\mathcal{M}_G) \hookrightarrow \text{Pic}_{L_X G}(\mathcal{Q}_G).$$

PROPOSITION 6.2. – *The forgetful morphism $\text{Pic}_{L_X G}(\mathcal{Q}_G) \rightarrow \text{Pic}(\mathcal{Q}_G)$ is injective.*

Proof. – The kernel of this morphism consists of the $L_X G$ -linearizations of the trivial bundle. Any two such trivializations differ by an automorphism of $pr_2^* \mathcal{O}_{\mathcal{Q}_G}$ that is by an invertible function on $L_X G \times \mathcal{Q}_G$. Since \mathcal{Q}_G is integral (Prop. 4.6), it is the direct limit of the integral projective varieties $\mathcal{Q}_G = \varinjlim \mathcal{Q}_G(N)_{\text{red}}$ and this function is the pull back of an invertible function f on $L_X G$. The cocycle conditions on the linearizations imply that f is a character, hence $f = 1$ by Lemma 5.2. ■

(6.3) By (6.1) and (6.2), the composition $\text{Pic}(\mathcal{M}_G) \rightarrow \text{Pic}(\mathcal{Q}_G)$ is injective. To prove surjectivity we should prove that the generator $\mathcal{O}_{\mathcal{Q}_G}(1)$ of $\text{Pic}(\mathcal{Q}_G)$ admits a $L_X G$ -linearization. In order to do this we will construct in the following sections a line bundle \mathcal{L} on \mathcal{M}_G that pulls back to $\mathcal{O}_{\mathcal{Q}_G}(1)$. Remark however the following:

LEMMA 6.4. – *The line bundle $\mathcal{O}_{\mathcal{Q}_G}(1)$ admits a $L_X G$ -linearization if and only if the restriction of the central extension (4.8.2) to $L_X G$ splits.*

Proof. – Let $\text{Mum}_{L_X G}(\mathcal{O}_{\mathcal{Q}_G}(1))$ be the Mumford group of $\mathcal{O}_{\mathcal{Q}_G}(1)$ with respect to $L_X G$. This is the group of pairs (f, g) with $g \in L_X G$ and $f : g^* \mathcal{O}_{\mathcal{Q}_G}(1) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Q}_G}(1)$. As \mathcal{Q}_G is direct limit of integral projective schemes, we get a central extension

$$(6.4.1) \quad 1 \longrightarrow G_m \longrightarrow \text{Mum}_{L_X G}(\mathcal{O}_{\mathcal{Q}_G}(1)) \longrightarrow L_X G \longrightarrow 1$$

In this setup, a $L_X G$ -linearization of $\mathcal{O}_{\mathcal{Q}_G}(1)$ corresponds to a splitting of (6.4.1). By functoriality of the Mumford group and the observation that \widehat{LG} is $\text{Mum}_{LG}(\mathcal{O}_{\mathcal{Q}_G}(1))$, it follows that the extension (6.4.1) is the pullback to $L_X G$ of (4.8.2), which proves the lemma. ■

Remark that, if exists, the splitting must be unique. Indeed, two splittings differ by a character of $L_X G$ and there is only the trivial character (Corollary 5.2).

(6.5) Let \mathcal{F} be a family of vector bundles of rank r with trivial determinant, parameterized by the locally noetherian \mathbb{C} -scheme S . Recall that the complex $Rpr_{1*}(\mathcal{F})$ may be represented by a perfect complex of length one K^\bullet and define $\mathcal{D}_{\mathcal{F}}$ to be $\det(K^\bullet)^{-1}$ [14]. This does not, up to canonical isomorphism, depend on the choice of K^\bullet . As the formation of the determinant commutes with base change, the fiber of $\mathcal{D}_{\mathcal{F}}$ over the point $s \in S$ is $\Lambda^{\max} H^0(X, \mathcal{F}(s))^* \otimes \Lambda^{\max} H^1(X, \mathcal{F}(s))$. The line bundle $\mathcal{D}_{\mathcal{F}}$ is called the *determinant of cohomology* line bundle associated to the family \mathcal{F} .

Let \mathcal{U} be the universal vector bundle on $\mathcal{M}_{SL_r} \times X$ and define the determinant line bundle $\mathcal{D} = \det(Rpr_{1*}\mathcal{U})^{-1}$. It has the following universal property: for every family \mathcal{F} of vector bundles parameterized by the locally noetherian \mathbb{C} -scheme S , we have $f_{\mathcal{F}}^*(\mathcal{D}) = \mathcal{D}_{\mathcal{F}}$ in $\text{Pic}(S)$, where $f_{\mathcal{F}} : S \rightarrow \mathcal{M}_{SL_r}$ is the deduced modular morphism.

LEMMA 6.6. – Under $\pi : \mathcal{Q}_{SL_r} \rightarrow \mathcal{M}_{SL_r}$, the line bundle \mathcal{D} pulls back to $\mathcal{O}_{\mathcal{Q}_{SL_r}}(1)$

Proof. – We consider the morphism $\bar{\varphi}$ of (4.11):

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{C}}^1 & \xrightarrow{\bar{\varphi}} & \mathcal{Q}_{SL_r} \\ & \searrow \gamma & \downarrow \pi \\ & & \mathcal{M}_{SL_r} \end{array}$$

We get a family \mathcal{E} of SL_r -bundles parameterized by $\mathbb{P}_{\mathbb{C}}^1$ and, by the above, we have to show that the determinant of this family is $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(1)$. By definition of φ one sees that it is enough to treat the rank 2 case, in which this family is easily identified: if we think of \mathcal{Q}_{SL_2} as parameterizing special lattices (see section 2 of [3]) then $\mathcal{E}_{[a:c]}$ is defined by the inclusion

$$W = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} \cdot (\mathbb{C}[[z]] \oplus \mathbb{C}[[z]]) \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z)).$$

hence may be seen, as the lattice

$$V = z^{-1}\mathbb{C}[[z]] \oplus \mathbb{C}[[z]] \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z))$$

defines the rank 2-bundle $F = \mathcal{O}_X(p) \oplus \mathcal{O}_X$, via the inclusion $W \subset V$ as the kernel of the morphism $F \rightarrow \mathbb{C}_p$ which maps the local sections $(z^{-1}f, g)$ to $af(p) - cg(p)$. Using this description of the family \mathcal{E} , we see, as in ([2], 3.4) that $\mathcal{D}_{\mathcal{E}} = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(1)$. ■

In particular, it follows from (6.1), (6.2) and (4.11) that $\text{Pic}(\mathcal{M}_{SL_r}) = \mathbb{Z}\mathcal{D}$.

(6.7) Consider a representation $\rho : G \rightarrow SL_r$ and the morphism obtained by extension of structure group $f_{\rho} : \mathcal{M}_G \rightarrow \mathcal{M}_{SL_r}$. Define the determinant of cohomology associated

to ρ by $\mathcal{D}_\rho = f_\rho^*(\mathcal{D})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_G & \longrightarrow & \mathcal{Q}_{SL_r} \\ & \searrow \phi_\rho & \downarrow \\ \mathcal{M}_G & \longrightarrow & \mathcal{M}_{SL_r} \end{array}$$

LEMMA 6.8. – (Kumar-Narasimhan-Ramanathan) Let d_ρ be the Dynkin index of ρ . Then the pullback of the determinant bundle under ϕ_ρ is $\mathcal{O}_{\mathcal{Q}_G}(d_\rho)$

Proof. – Consider the pullback diagram of (4.8.2) for LSL_r :

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_m & \longrightarrow & \widetilde{LG} & \longrightarrow & Lg & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow L\rho & & \\ 1 & \longrightarrow & G_m & \longrightarrow & \widetilde{LSL_r} & \longrightarrow & LSL_r & \longrightarrow & 1 \end{array}$$

As $\widetilde{LSL_r}$ is $\text{Mum}_{LSL_r}(\mathcal{O}_{\mathcal{Q}_{SL_r}}(1))$ (cf. 6.4), by functoriality of the Mumford group, we have $\widetilde{LG} = \text{Mum}_{LG}(\phi_\rho^* \mathcal{O}_{\mathcal{Q}_{SL_r}}(1))$. Looking at the differentials, on the level of Lie algebras, we restrict the universal central extension of $L\mathfrak{sl}_r$ to $L\mathfrak{g}$ hence \widetilde{LG} is the extension of (2.3), so $\phi_\rho^* \mathcal{O}_{\mathcal{Q}_{SL_r}}(1)$ has to be $\mathcal{O}_{\mathcal{Q}_G}(d_\rho)$. ■

COROLLARY 6.9. – As a pullback, the line bundle $\mathcal{O}_{\mathcal{Q}_G}(d_\rho)$ is $L_X G$ -linearized.

(6.10) By the above, for series A and C , as the Dynkin index of the standard representation is 1, all line bundles on \mathcal{Q}_G are $L_X G$ -linearized. For series B and D (and also for G_2) the Dynkin index of the standard representation, which factors through SO_r (resp. SO_7), is 2. We will construct in Section 7 a canonical (once a theta-characteristic κ on X is fixed) square root \mathcal{P}_κ of \mathcal{D}_{ω_1} on \mathcal{M}_{SO_r} . The pullback of \mathcal{P}_κ to \mathcal{M}_G , which does not depend on the choice of κ by (6.1) and (6.2), will pull back to $\mathcal{O}_{\mathcal{Q}_G}(1)$. Therefore, for classical G and G_2 , all line bundles on \mathcal{Q}_G are $L_X G$ -linearized and Theorem 1.1 (for $n = 0$) follows from (6.1), (6.2) and (4.11).

7. Pfaffians

Let k be an algebraically closed field of characteristic $\neq 2$ and S a k -scheme.

(7.1) The Picard categories

Let A be \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$. Denote by \mathfrak{L}_A the groupoid of A -graded invertible \mathcal{O}_S -modules. The objects of \mathfrak{L}_A are pairs $[L] = (L, a)$ of invertible \mathcal{O}_S -modules L and locally constant functions $a : S \rightarrow A$, morphisms $[f] : [L] \rightarrow [M]$ are defined if $a = b$ and are isomorphisms $f : L \rightarrow M$ of \mathcal{O}_S -modules. Denote $\mathbb{1}_A$ the object $(\mathcal{O}_S, 0)$. The category \mathfrak{L}_A has tensor products, defined by $[L] \otimes [M] = (L \otimes M, a + b)$. Given $[L]$ and $[M]$ we have Koszul's symmetry isomorphism $\sigma_{[L],[M]} : [L] \otimes [M] \rightarrow [M] \otimes [L]$ defined on local sections ℓ and m by $\sigma_{L,M}(\ell \otimes m) = (-1)^{ab} m \otimes \ell$.

Denote \det_A the functor from the category of coherent locally free \mathcal{O}_S -modules with isomorphisms defined by $\det_A = (\Lambda^{\max}, \text{rang}(V))$ and $\det_A(f) = \Lambda^{\max}(f)$.

In the following we drop the subscript A for $A = \mathbb{Z}$ and replace it by 2 for $A = \mathbb{Z}/2\mathbb{Z}$.

(7.2) Pfaffians

Let V be a coherent locally free \mathcal{O}_S -module of rank $2n$. Let $\text{pf} : \Lambda^2 V^* \rightarrow \Lambda^{2n} V^*$ be defined by $\omega \mapsto \omega^n$. Suppose $\alpha : V \rightarrow V^*$ is skewsymmetric. View α as a section of $\Lambda^2 V^*$ and define the *pfaffian* of α as $\text{pf}(\alpha) : \mathcal{O}_S \rightarrow \Lambda^{2n} V^*$. Denote by $\det(\alpha)$ the map $\Lambda^{2n} V \rightarrow \Lambda^{2n} V^*$ so $\det(\alpha)$ may be seen as a section $\mathcal{O}_S \rightarrow \Lambda^{2n} V^* \otimes_{\mathcal{O}_S} \Lambda^{2n} V^*$. Then we have

$$(7.2.1) \quad \det(\alpha) = \text{pf}(\alpha)^2$$

and moreover, if u is an endomorphism of V^* ,

$$(7.2.2) \quad \text{pf}(u\alpha u^*) = \det(u)\text{pf}(\alpha)$$

(7.3) The pfaffian functor

We consider the following category $\mathcal{A} = \mathcal{A}^\bullet(S)$: objects are complexes of locally free coherent \mathcal{O}_S -modules concentrated in degrees 0 and 1 of the form

$$0 \longrightarrow E \xrightarrow{\alpha} E^* \longrightarrow 0$$

with α skewsymmetric. Morphisms between two such complexes E^\bullet and F^\bullet are morphisms of complexes $f^\bullet : E^\bullet \rightarrow F^\bullet$ such that $f^{\bullet\bullet}[-1]$ is a homotopy inverse of f^\bullet , i.e. $f^{\bullet\bullet}[-1] \circ f^\bullet$ and $f^\bullet \circ f^{\bullet\bullet}[-1]$ are homotopic to the identity. Let $\pi : \mathcal{L} \rightarrow \mathcal{L}_2$ be the projection functor, $\Delta : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ be the functor defined by $[L] \mapsto [L] \otimes [L]$ and $[f] \mapsto [f] \otimes [f]$ and $\text{Det} : \mathcal{A} \rightarrow \mathcal{L}$ be the determinant functor [14].

PROPOSITION 7.4. – *There is a natural functor, $\text{Pf} : \mathcal{A}^\bullet \rightarrow \mathcal{L}_2$, commuting with base changes, and a natural isomorphism of functors:*

$$\pi \circ \text{Det} \xrightarrow{\sim} \Delta \circ \text{Pf}.$$

Moreover, if $f^\bullet : E^\bullet \rightarrow F^\bullet$ is homotopic to the identity then $\text{Pf}(f^\bullet) = \text{id}$.

Proof. – Define Pf on the level of objects by $\text{Pf}(E^\bullet) = \det_2(E)$ (cf. 7.1). On the level of morphisms we do the following. Let $(f_0, f_1) : E^\bullet \xrightarrow{f^\bullet} F^\bullet$ be a morphism of \mathcal{A}^\bullet :

$$\begin{array}{ccc} E & \xrightarrow{\alpha_E} & E^* \\ f_0 \downarrow & & \downarrow f_1 \\ F & \xrightarrow{\alpha_F} & F^* \end{array}$$

and consider the complex C_f^\bullet (which is up to sign the cone of f^\bullet)

$$C_f^\bullet = 0 \longrightarrow E \xrightarrow{\begin{pmatrix} \alpha_E \\ -f_0 \end{pmatrix}} E^* \oplus F \xrightarrow{(f_1 \quad \alpha_F)} F^* \longrightarrow 0$$

As f^\bullet is a quasi-isomorphism, C_f^\bullet is acyclic. By the usual additivity property of determinants, we get a canonical isomorphism

$$d(f) : \Lambda^{\max} E \otimes \Lambda^{\max} F^* \rightarrow \Lambda^{\max} E^* \otimes \Lambda^{\max} F.$$

Recall that this isomorphism is defined by taking a section $\begin{pmatrix} u \\ v \end{pmatrix}$ of $(f_1 \alpha_F)$ and calculating the determinant, which is independent of this choice, of the morphism

$$M(f) = \begin{pmatrix} \alpha & u \\ -f_0 & v \end{pmatrix} \in \text{Hom}(E \oplus F^*, E^* \oplus F)$$

LEMMA 7.5. – *There is a skew-symmetric morphism $\gamma_f \in \text{Hom}(F^*, F)$ such that $\begin{pmatrix} f_0^* \\ \gamma \end{pmatrix}$ is a section of $(f_1 \alpha_F)$.*

Proof. – As $f \circ f^*[-1]$ is homotopic to the identity there is a morphism h such that $f_0 f_1^* - 1 = h \alpha_F$ and $f_1 f_0^* - 1 = \alpha_F h$. Now define $\gamma_f = \frac{h^* - h}{2}$. ■

The pfaffian of the skew-symmetric morphism

$$M(f, \gamma_f) = \begin{pmatrix} \alpha & f_0^* \\ -f_0 & \gamma_f \end{pmatrix} \in \text{Hom}(E \oplus F^*, E^* \oplus F)$$

defines a section $\text{pf}(M(f, \gamma_f)) : \mathcal{O}_S \rightarrow \Lambda^{\max} E^* \otimes \Lambda^{\max} F$.

LEMMA 7.6. – *The section $\text{pf}(M(f, \gamma_f))$ is independent of the choice of γ_f .*

Proof. – Suppose γ'_f is another morphism as in (7.5). Then there is $g \in \text{Hom}(F^*, E)$ such that $\alpha_E g = 0$ and $f_0 g = -g^* f_0^*$ [use that γ_f and γ'_f are skew]. These relations give

$$M(f, \gamma'_f) = \begin{pmatrix} 1 & 0 \\ \frac{g}{2} & 1 \end{pmatrix} M(f, \gamma_f) \begin{pmatrix} 1 & \frac{g}{2} \\ 0 & 1 \end{pmatrix}$$

which in turn implies the required equality by (7.2.2). ■

As $\text{rank}(E) = \text{rank}(F) \bmod 2$, we get the isomorphism in \mathcal{L}_2 :

$$\text{pf}(M(f)) : \mathbb{I}_2 \xrightarrow{\sim} \det_2(E)^* \otimes \det_2(F).$$

Define the pfaffian of f^\bullet by

$$\text{Pf}(f^\bullet) : \det_2(E) \xrightarrow{1 \otimes \text{pf}(M(f, \gamma_f))} \det_2(E) \otimes \det_2(E)^* \otimes \det_2(F) \xrightarrow{\text{ev}_{\det_2(E)}} \det_2(F)$$

LEMMA 7.7. – $\text{Pf} : \mathcal{A} \rightarrow \mathcal{L}_2$ defines a functor.

Proof. – As $\text{pf}(M(\text{Id}, 0)) = 1$, we have $\text{Pf}(\text{Id}) = \text{Id}$. Let $f^\bullet : E^\bullet \rightarrow F^\bullet$ and $g : F^\bullet \rightarrow G^\bullet$ be two morphisms of \mathcal{A} . Then the following diagram is commutative

$$\begin{array}{ccc} \mathbb{I}_2 & \xrightarrow{\text{pf}(M(f, \gamma_f)) \otimes \text{pf}(M(g, \gamma_g))} & \det_2(E)^* \otimes \det_2(F) \otimes \det_2(F)^* \otimes \det_2(G) \\ & \searrow \text{pf}(M(g \circ f, \gamma_{g \circ f})) & \downarrow 1 \otimes \text{ev}_{\det_2(F)} \otimes 1 \\ & & \det_2(E)^* \otimes \det_2(G) \end{array}$$

Indeed, remark that $\gamma_{gf} = g_0\gamma_f g_0^* + \gamma_g$ satisfies (7.5) for $g \circ f$ and make use of (7.2.2) first with

$$\begin{pmatrix} \alpha_E & f_0^* & 0 & f_0^* g_0^* \\ -f_0 & \gamma_f & 1 & \gamma_f g_0^* \\ 0 & -1 & 0 & 0 \\ -g_0 f_0 & g_0 \gamma_f & 0 & \gamma_{gf} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f_1 & -\alpha_F & 1 & 0 \\ 0 & g_0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_E & f_0^* & 0 & 0 \\ -f_0 & \gamma_f & 0 & 0 \\ 0 & 0 & \alpha_F & g_0^* \\ 0 & 0 & -g_0 & \gamma_g \end{pmatrix} \begin{pmatrix} 1 & 0 & -f_1^* & 0 \\ 0 & 1 & \alpha_F & g_0^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and then with

$$\begin{pmatrix} \alpha_E & 0 & 0 & f_0^* g_0^* \\ 0 & \gamma_f & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -g_0 f_0 & 0 & 0 & \gamma_{gf} \end{pmatrix} = \begin{pmatrix} 1 & 0 & f_0^* & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & g_0 \gamma_f & 1 \end{pmatrix} \begin{pmatrix} \alpha_E & f_0^* & 0 & f_0^* g_0^* \\ -f_0 & \gamma_f & 1 & \gamma_f g_0^* \\ 0 & -1 & 0 & 0 \\ -g_0 f_0 & g_0 \gamma_f & 0 & \gamma_{gf} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f_0 & 0 & 1 & -\gamma_f g_0^* \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The commutativity of the above diagram shows $\text{Pf}(g \circ f) = \text{Pf}(g) \circ \text{Pf}(f)$. ■

The statement on the natural transformation follows from the definitions and (7.2.1). It remains to prove that if $f^\bullet : E^\bullet \rightarrow E^\bullet$ is homotopic to the identity, then $\text{Pf}(f) = \text{Id}$. Indeed, let $h : E^* \rightarrow E$ be such that $f_0 - h\alpha_E = 1$ and $f_1 - \alpha h = 1$. Then $\gamma_f := -h + f_0 h^*$ satisfies (7.5) and the statement follows from (7.2.2) and

$$\begin{pmatrix} a_E & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \begin{pmatrix} a_E & f_0^* \\ -f_0^* & \gamma_f \end{pmatrix} \begin{pmatrix} 1 & h^* \\ 0 & 1 \end{pmatrix}$$

This completes the proof of Proposition 7.4. ■

(7.8) The Pfaffian bundle

Consider $G = \text{Spin}_r$ with $r \geq 3$ (resp. $G = G_2$). Then the standard representation ϖ_1 factors through SO_r (resp. SO_7). The stack \mathcal{M}_{SO_r} has two components: $\mathcal{M}_{SO_r}^0$ and $\mathcal{M}_{SO_r}^1$. They are distinguished by the second Stiefel-Whitney class

$$w_2 : H_{\text{ét}}^1(X, SO_r) \rightarrow H_{\text{ét}}^2(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

Let κ be a theta-characteristic on X . Twisting by κ , we may and will see a SO_r -bundle as a vector bundle F with trivial determinant together with a *symmetric* isomorphism $\sigma : F \rightarrow F^\vee$, where $F^\vee = \underline{\text{Hom}}_{\mathcal{O}_X}(F, \omega_X)$. The following Proposition shows the existence, for every κ , of a canonical square root \mathcal{P}_κ of the determinant bundle \mathcal{D}_{ϖ_1} over \mathcal{M}_{SO_r} .

PROPOSITION 7.9. – *Let (\mathcal{F}, σ) be a family of vector bundles \mathcal{F} equipped with a quadratic form σ with values in ω_X parameterized by the locally noetherian k -scheme S . Then the determinant of cohomology $\mathcal{D}_{\mathcal{F}}$ admits a canonical square root $\mathcal{P}_{(\mathcal{F}, \sigma)}$. Moreover, if $f : S' \rightarrow S$ is a morphism of locally noetherian k -schemes then we have $\mathcal{P}_{(f^* \mathcal{F}, f^* \sigma)} = f^* \mathcal{P}_{(\mathcal{F}, \sigma)}$.*

Proof. – We will define $\mathcal{P}_{(\mathcal{F}, \sigma)}$ first locally and then construct patching data. By ([28], prop. 2.1 and proof of Corollary 2.2, cf. also [13]), Zariski locally on S , there are length

1 complexes M^\bullet of finite free \mathcal{O}_S -modules and quasi-isomorphisms $f : M^\bullet \rightarrow Rpr_{1*}(\mathcal{F})$ such that the composition in the derived category $D(S)$ (use σ and Grothendieck duality)

$$M^\bullet \xrightarrow{f} Rpr_{1*}(\mathcal{F}) \xrightarrow{\tau} R\mathcal{H}om^\bullet(Rpr_{1*}(\mathcal{F}), \mathcal{O}_S)[-1] \xrightarrow{f^*[-1]} M^{\bullet*}[-1]$$

lifts to a symmetric isomorphism of complexes $\varphi : M^\bullet \rightarrow M^{\bullet*}[-1]$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^0 & \xrightarrow{d_{M^\bullet}} & M^1 & \longrightarrow & 0 \\ & & \varphi_0 \downarrow \wr & \searrow \alpha & \wr \downarrow \varphi_0^* & & \\ 0 & \longrightarrow & M^{1*} & \xrightarrow{-d_{M^\bullet}^*} & M^{0*} & \longrightarrow & 0 \end{array}$$

Define \widetilde{M}^\bullet by $0 \rightarrow M^0 \xrightarrow{\alpha} M^{0*} \rightarrow 0$. Then α is skew and we have a natural isomorphism of complexes $\psi : M^\bullet \rightarrow \widetilde{M}^\bullet$ such that $\psi^*[-1]\psi = f^*[-1]\tau f$ in $D^b(S)$.

Now cover S by open subsets U_i together with complexes $(M_i^\bullet, d_{M_i^\bullet})$ and quasi-isomorphisms $f_i : M_i^\bullet \rightarrow Rpr_{1*}(\mathcal{F})|_{U_i}$ as above. We define $\mathcal{P}_{(\mathcal{F}, \sigma)}$ over U_i by $\mathcal{P}_{i, (\mathcal{F}, \sigma)} = \text{Pf}(\widetilde{M}_i^\bullet)$ and construct patching data $\rho_{ij} : \mathcal{P}_{i, (\mathcal{F}, \sigma)} \xrightarrow{\sim} \mathcal{P}_{j, (\mathcal{F}, \sigma)}$ over $U_{ij} = U_i \cap U_j$ in the following way. Define first the morphism of complexes $\Sigma_{ij} : \widetilde{M}_i \rightarrow \widetilde{M}_j$ as a lifting of the isomorphism in $D^b(U_{ij})$

$$\psi_j^{*-1}[-1]f_j^*[-1]\tau f_i\psi_i^{-1},$$

then ρ_{ij} by $\text{Pf}(\Sigma_{ij})$ (note that it follows from the symmetry of σ and that the components of the \widetilde{M}_i are free that Σ_{ij} is a morphism of $\mathcal{A}(U_{ij})$). By (7.4), ρ_{ij} does not depend on the particular chosen lifting and the functoriality of Pf translates into $\rho_{ii} = \text{Id}$, $\rho_{ij} = \rho_{ik}\rho_{kj}$ and also $\rho_{ij} = \rho_{ji}^{-1}$, hence the $\mathcal{P}_{i, (\mathcal{F}, \sigma)}$ glue together to our requested line bundle $\mathcal{P}_{(\mathcal{F}, \sigma)}$. The construction is functorial, so it remains to check that we have constructed a square root of the determinant bundle: but this follows again from Proposition 7.4, as over U_i we have $\mathcal{P}_{i, (\mathcal{F}, \sigma)} \otimes \mathcal{P}_{i, (\mathcal{F}, \sigma)} = \det(\widetilde{M}_i^\bullet)$ and as, by [14], the $\det(\widetilde{M}_i^\bullet)$ path together (via $f_i\psi_i^{-1}$), to $\mathcal{D}_{\mathcal{F}}$. ■

Considering the universal family over $\mathcal{M}_{SO_r} \times X$, we get, by the above, for every theta-characteristic κ a line bundle \mathcal{P}_κ over \mathcal{M}_{SO_r} .

(7.10) The pfaffian divisor

Let $r \geq 3$ and (\mathcal{E}, q) be the universal quadratic bundle over $\mathcal{M}_{SO_r}^0 \times X$. For κ a theta-characteristic, let us denote by Θ_κ the substack defined by

$$\Theta_\kappa = \text{div}(Rpr_{1*}(\mathcal{E} \otimes pr_2^*\kappa)).$$

CLAIM. – This substack is a divisor if and only if r or κ are even.

Proof. – Let $P = (E, q)$ be a SO_r -bundle, $r \geq 3$ and κ be a theta-characteristic. Then

$$(7.10.1) \quad w_2(P) = h^0(E \otimes \kappa) + \text{rh}^0(\kappa) \bmod 2.$$

Indeed, by Riemann's invariance mod 2 theorem, the right hand side of (7.10.1), denoted $w'_2(P)$ in the following, is constant over the 2 connected components of \mathcal{M}_{SO_r} . Because

(7.10.1) is true at the trivial SO_r -bundle \mathcal{T} , it is enough to prove that w'_2 is not constant. Let $L, M \in J_2$ (where $J_2 =$ points of order 2 of the jacobian) such that for the Weil pairing we have $\langle L, M \rangle = 1$. The choice of a trivialization of their squares defines a non degenerated quadratic form on $E = (L \otimes M) \oplus L \oplus M \oplus (r-3)\mathcal{O}_X$ hence a SO_r -bundle P . By [Mu], we know that we have $w'_2(P) = \langle L, M \rangle \neq 0 = w'_2(\mathcal{T})$, which proves (7.10.1).

Now choose an ineffective theta-characteristic κ_0 and set $L = \kappa_0 \otimes \kappa^{-1}$. If r is even, there exists a SO_r -bundle $P = (E, q)$ such that $H^0(E \otimes \kappa) = 0$ and $w_2(P) = 0$ (choose $E = rL$ with the obvious quadratic form and use (7.10.1)). If r is odd and κ is even, there exists a SO_r -bundle $P = (E, q)$ such that $H^0(E \otimes \kappa) = 0$ and $w_2(P) = 0$ (by Lemma 1.5 of [2], there is a SL_2 -bundle F on X such that $H^0(X, \text{ad}(F) \otimes \kappa) = 0$, then choose $E = \text{ad}(F) \oplus (r-3)L$ with the obvious quadratic form.) If r and κ are odd, then $H^0(E \otimes \kappa)$ is odd for all $P \in \mathcal{M}_{SO_r}^0$. ■

As the perfect complex $Rpr_{1*}(\mathcal{E} \otimes pr_2^* \kappa)$ can be locally represented by a skew-symmetric perfect complex of length one $L \rightarrow^\alpha L^*$ (cf. Proof of 7.9), the pfaffian of α defines (if r or κ is even) a local equation of an effective divisor $\tilde{\Theta}_\kappa$ such that $2\tilde{\Theta}_\kappa = \Theta_\kappa$. This gives an easier way to define, by smoothness of \mathcal{M}_G , the pfaffian line bundle. The reason which motivated our construction above was to define this square root for arbitrary quadratic bundles (not only the even ones) and to make a construction for all theta-characteristics and not only the even ones (when r is odd).

(7.11) *Proof of (1.5).* – The dualizing line bundle $\omega_{\mathcal{M}_G}$ on \mathcal{M}_G is by definition the determinant line bundle of the cotangent complex of \mathcal{M}_G . Let $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ be the adjoint representation. Then $\omega_{\mathcal{M}_G} = \mathcal{D}_{\text{Ad}}^{-1}$. Suppose that G is semi-simple. Then the adjoint representation factors through the special orthogonal group because of the existence of the Cartan-Killing form. Choose a theta-characteristic κ on X . Then, as in (7.8), we can define a square root $\omega_{\mathcal{M}_G}^{\frac{1}{2}}(\kappa)$ of $\omega_{\mathcal{M}_G}$.

8. Proof of theorems 1.1 and 1.2

Throughout this section G is simple, simply connected over \mathbb{C} . We will now prove theorems 1.1 and 1.2. We will start by defining the stack of quasi-parabolic G -bundles.

(8.1) We use the notations of Section 2. We will recall some standard facts for Lie groups, which we will use later. Let G be the simple and simply connected algebraic group associated to \mathfrak{g} . Denote by $T \subset G$ the Cartan subgroup associated to $\mathfrak{h} \subset \mathfrak{g}$ and by $B \subset G$ the Borel subgroup associated to $\mathfrak{b} \subset \mathfrak{g}$. Given a subset Σ of the set of simple roots Π (nodes of the Dynkin diagram), we can define a subalgebra $\mathfrak{p}_\Sigma = \mathfrak{b} \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_{-\alpha}) \subset \mathfrak{g}$, hence a subgroup $P_\Sigma \subset G$. Remark that $P_\emptyset = B$, $P_\Pi = G$ and that all P_Σ contain B . The subgroup P_Σ is parabolic and conversely any *standard* (i.e. containing B) parabolic subgroup arises in this way. Fix $\Sigma \subset \Pi$ and let $\Gamma = \Pi - \Sigma$. Denote by $X(P_\Sigma)$ the character group of P_Σ . Any weight λ such that $\lambda(H_\alpha) = 0$ for all $\alpha \in \Sigma$ defines, via the exponential map, a character of P_Σ and all characters arise in this way, i.e. $X(P_\Sigma) = \{\lambda \in P / \lambda(H_\alpha) = 0 \text{ for all } \alpha \in \Sigma\}$. In particular, the choice of a dominant weight λ defines a standard parabolic subgroup (i.e. P_Σ with $\Sigma = \{\alpha / \lambda(H_\alpha) = 0\}$) and a dominant character (with respect to B) of P and vice versa.

Given $\chi \in X(P_\Sigma)$ we can define the line bundle $L_\chi = G \times^{P_\Sigma} \mathbb{C}_\chi$ on the homogeneous space G/P_Σ . In general, there is an exact sequence ([11], prop. 3.1)

$$1 \longrightarrow X(G) \longrightarrow X(P_\Sigma) \longrightarrow \text{Pic}(G/P_\Sigma) \longrightarrow \text{Pic}(G) \longrightarrow \text{Pic}(P) \longrightarrow 0.$$

As G is simple, we have $X(G) = 0$ and as G is simply connected, we have $\text{Pic}(G) = 0$ ([11], Cor. 4.5). We get the isomorphism $X(P_\Sigma) \xrightarrow{\sim} \text{Pic}(G/P_\Sigma)$. In particular, the Picard group of G/P_Σ is isomorphic to the free abelian group generated over Γ .

(8.2) Consider closed points p_1, \dots, p_n of X , labeled with standard parabolic subgroup P_1, \dots, P_n . Let $\Sigma_1, \dots, \Sigma_n$ be the associated subsets of simple roots and $\Gamma_i = \Pi - \Sigma_i$ for $i \in \{1, \dots, n\}$. In the following, underlining a character will mean that we consider the associated sequence, e.g. \underline{P} will denote the sequence (P_1, \dots, P_n) , etc. Let E be a G -bundle. As G acts on G/P_i we can define the associated G/P_i -bundle $E(G/P_i)$.

DEFINITION 8.3. – (cf. [21]) A quasi-parabolic G -bundle of type \underline{P} is a G -bundle E on X together with, for all $i \in \{1, \dots, n\}$, an element $F_i \in E(G/P_i)(p_i)$. A parabolic G -bundle of type $(\underline{P}, \underline{m})$ is a quasi-parabolic G -bundle of type \underline{P} together with, for $i \in \{1, \dots, n\}$, parabolic weights $(m_{i,j})_{j \in \Gamma_i}$ where the $m_{i,j}$ are strictly positive integers.

(8.4) Let R be a \mathbb{C} -algebra, $S = \text{Spec}(R)$. A family of quasi-parabolic G -bundles of type \underline{P} parameterized by S is a G -bundle E over $S \times X$ together with n sections $\sigma_i : S \rightarrow E(G/P_i)|_{S \times \{p_i\}}$. A morphism from $(E, \underline{\sigma})$ to $(E', \underline{\sigma}')$ is a morphism $f : E \rightarrow E'$ of G -bundles such that for all $i \in \{1, \dots, n\}$ we have $\sigma'_i = f|_{S \times \{p_i\}} \circ \sigma_i$. We get a pseudo-functor from the category of \mathbb{C} -algebras to the category of groupoids by associating to the \mathbb{C} -algebra R the groupoid having as objects families of quasi-parabolic G -bundles of type \underline{P} parameterized by $S = \text{Spec}(R)$ and as arrows isomorphisms between such families. Moreover for any morphism $R \rightarrow R'$ we have a natural functor between the associated groupoids. This defines the \mathbb{C} -stack of quasi-parabolic G -bundles of type \underline{P} which we will denote by $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$. The stack $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$ has, as \mathcal{M}_G , a natural interpretation as a double quotient stack. Define

$$\mathcal{Q}_G^{par}(\underline{p}, \underline{P}) = \mathcal{Q}_G \times \prod_{i=1}^n G/P_i.$$

The ind-group $L_X G$ acts on \mathcal{Q}_G and, by evaluation $ev(p_i) : L_X G \rightarrow G$ at p_i , also on each factor G/P_i . We get a natural action of $L_X G$ on $\mathcal{Q}_G^{par}(\underline{p}, \underline{P})$. The analogue of Theorem 1.3 for quasi-parabolic G -bundles is

THEOREM 8.5. – (Uniformization) There is a canonical isomorphism of stacks

$$\bar{\pi} : L_X G \backslash \mathcal{Q}_G^{par}(\underline{p}, \underline{P}) \xrightarrow{\sim} \mathcal{M}_G^{par}(\underline{p}, \underline{P}).$$

Moreover the projection map is locally trivial for the étale topology.

Proof. – Let R be a \mathbb{C} -algebra, $S = \text{Spec}(R)$. To an element (E, ρ, f) of $\mathcal{Q}_G^{par}(\underline{p}, \underline{P})(R)$ (with $f_i \in \text{Mor}(S, G/P_i)$), we can associate a family of quasi-parabolic G -bundles of type \underline{P} parameterized by S in the following way. We only have to define the sections:

$$\sigma_i : S \xrightarrow{(\text{id}, f_i)} S \times G/P_i \xrightarrow{\rho_i(G/P_i)} E(G/P_i)|_{S \times \{p_i\}}.$$

We get a $L_X G$ -equivariant map $\pi : \mathcal{Q}_G^{par}(\underline{p}, \underline{P}) \rightarrow \mathcal{M}_G^{par}(\underline{p}, \underline{P})$ which induces the map on the level of stacks $\bar{\pi} : L_X G \backslash \mathcal{Q}_G^{par}(\underline{p}, \underline{P}) \rightarrow \mathcal{M}_G^{par}(\underline{p}, \underline{P})$. Conversely, let $(E, \underline{\sigma})$ be a family of quasi-parabolic G -bundles of type \underline{P} parameterized by $S = \text{Spec}(R)$. For any R -algebra R' , let $T(R')$ be the set of trivializations ρ of $E_{R'}$ over $X_{R'}$. This defines a R -space T which by Theorem 3.11 is a $L_X G$ -bundle. To any element in $T(R')$, we can associate the family \underline{f} by

$$f_i : S \xrightarrow{\sigma_i} E(G/P_i)_{|S \times \{p_i\}} \xrightarrow{\rho_i(G/P_i)^{-1}} S \times G/P_i \xrightarrow{pr_2} G/P_i.$$

In this way we associate functorially to the objects $(E, \underline{\sigma})$ of $\mathcal{M}_G^{par}(\underline{p}, \underline{P})(R)$ $L_X G$ -equivariant maps $\alpha : T \rightarrow \mathcal{Q}_G^{par}(\underline{p}, \underline{P})$. This defines a morphism of stacks

$$\mathcal{M}_G^{par}(\underline{p}, \underline{P}) \longrightarrow L_X G \backslash \mathcal{Q}_G^{par}(\underline{p}, \underline{P})$$

which is the inverse of $\bar{\pi}$. The second statement is clear from the proof of Theorem 1.3. \blacksquare

(8.6) We study first line bundles over $\mathcal{Q}_G^{par}(\underline{p}, \underline{P})$. Using (4.11), (8.1) and $H^1(G/P_i, \mathcal{O}) = 0$, we obtain the following proposition, proving, as $L_X G$ has no characters (Corollary 5.2), Theorem 1.1.

PROPOSITION 8.7. – *We have*

$$\text{Pic}(\mathcal{Q}_G^{par}(\underline{p}, \underline{P})) = \mathbb{Z}\mathcal{O}_{\mathcal{Q}_G}(1) \times \prod_{i=1}^n \text{Pic}(G/P_i) = \mathbb{Z}\mathcal{O}_{\mathcal{Q}_G}(1) \times \prod_{i=1}^n X(P_i).$$

(8.8) Let $(E, \underline{\sigma})$ be a family of quasi-parabolic G -bundles of type \underline{P} parameterized by the \mathbb{C} -scheme $S = \text{Spec}(R)$. Fix $i \in \{1, \dots, n\}$ and $j \in \Gamma_i$. We may view $E \rightarrow E(G/P_i)$ as a P_i -bundle. Therefore the character of P_i defined by $-\varpi_j$ defines a line bundle on $E(G/P_i)$, hence by pullback, using the section $\sigma_i : S \rightarrow E(G/P_i)_{|S \times \{p_i\}}$, a line bundle $\mathcal{L}_{i,j}$ over S . This works for any S and we get a line bundle over the stack $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$ which we denote again by $\mathcal{L}_{i,j}$.

(8.9) Conformal blocks and generalized theta functions

We now prove Theorem 1.2. We use the notations of section 2 and start by defining the space of conformal blocks. Fix an integer $\ell \geq 0$ (the level) and let p_1, \dots, p_n be distinct closed points of X (we allow $n = 0$ i.e. no points), each of it labeled with a dominant weight λ_i lying in the fundamental alcove P_ℓ . Choose also another point $p \in X$, distinct from the points p_1, \dots, p_n . Define

$$\mathcal{H}_\lambda = \mathcal{H}(\ell) \otimes (\otimes_{i=1}^n L_{\lambda_i}).$$

We can map $L_X \mathfrak{g}$ via the Laurent developpement at the point p to $L\mathfrak{g}$. The restriction to $L_X \mathfrak{g}$ of the universal central extension $\widehat{L\mathfrak{g}}$ of $L\mathfrak{g}$ splits by the residue theorem, hence $L_X \mathfrak{g}$ may be considered as a *sub Lie-algebra* of $\widehat{L\mathfrak{g}}$. In particular, $\mathcal{H}(\ell)$ is a $L_X \mathfrak{g}$ -module. Evaluating $X \otimes f \in L_X \mathfrak{g}$ at the point p_i , we may consider L_{λ_i} as a $L_X \mathfrak{g}$ -module. Therefore \mathcal{H}_λ is a (left) $L_X \mathfrak{g}$ -module. Define the space of conformal blocks (or vacua) by

$$V_X(\underline{p}, \underline{\lambda}) = [\mathcal{H}_\lambda^*]^{L_X \mathfrak{g}} := \{\psi \in \mathcal{H}_\lambda^* / \psi.(X \otimes f) = 0 \ \forall X \otimes f \in L_X \mathfrak{g}\}.$$

This definition is Beauville's description [1] (*see* also [29]) of the space of conformal blocks of Tsuchiya, Ueno and Yamada [31]. The labeling of the points p_i induces $\Sigma_i = \{\alpha \in \Pi / \lambda_i(H_\alpha) = 0\}$, $\Gamma_i = \Pi - \Sigma_i$ and $m_{i,j} = \lambda_i(H_{\alpha_j})$ for $j \in \Gamma_i$, that is the type of a parabolic G -bundle. In particular, we get a natural line bundle on the moduli stack $\mathcal{M}_G^{par}(\underline{p}, \underline{P})$ defined by

$$\mathcal{L}(\ell, \underline{m}) = \mathcal{L}^\ell \boxtimes \left(\boxtimes_{i=1}^n \left(\boxtimes_{j \in \Gamma_i} \mathbb{Z} \mathcal{L}_{i,j}^{m_{i,j}} \right) \right).$$

By construction, for the pull back of $\mathcal{L}(\ell, \underline{m})$ to $\mathcal{Q}_G^{par}(\underline{p}, \underline{P})$ we have

$$\pi^* \mathcal{L}(\ell, \underline{m}) = \mathcal{O}_{\mathcal{Q}_G}(\ell) \boxtimes \left(\boxtimes_{i=1}^n \mathcal{L}_{-\lambda_i} \right)$$

where $\mathcal{L}_{-\lambda_i}$ is the line bundle on the homogeneous space G/P_i defined by the character corresponding to the weight $-\lambda_i$.

(8.10) *Proof of (1.2).* – We proceed in four steps.

Step 1. – As a pullback, $\pi^* \mathcal{L}(\ell, \underline{m})$ is canonically $L_X G$ -linearized, that is equipped with $\varphi : m^*(\pi^* \mathcal{L}(\ell, \underline{m})) \xrightarrow{\sim} pr_2^*(\pi^* \mathcal{L}(\ell, \underline{m}))$. Denote by $[H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))]^{L_X G}$ the space of $L_X G$ -invariant sections, that is the sections s such that $\varphi(m^* s) = pr_2^* s$. By Lemma 7.2 of [3] we have the canonical isomorphism

$$H^0(\mathcal{M}_G^{par}(\underline{p}, \underline{P}), \mathcal{L}(\ell, \underline{m})) \xrightarrow{\sim} [H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))]^{L_X G}$$

Denote by $[H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))]^{L_X \mathfrak{g}}$ the sections annihilated by $\text{Lie}(L_X G) = L_X \mathfrak{g}$. By Proposition 7.4 of [3], using that $L_X G$ and $\mathcal{Q}_G^{par}(\underline{p}, \underline{P})$ are integral (5.1) and Section 4), we have the canonical isomorphism

$$[H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))]^{L_X G} \xrightarrow{\sim} [H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))]^{L_X \mathfrak{g}}$$

Step 2. – By definition of \widehat{LG} , the space $H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))$ is naturally a \widehat{LG} -module. Moreover we know that \widehat{LG} splits over $L_X G$ (at least for classical G and G_2) and that this splitting is *unique*. The action of $L_X \mathfrak{g} \subset \widehat{L\mathfrak{g}}$ deduced from this inclusion on $H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^* \mathcal{L}(\ell, \underline{m}))$ is therefore the same as the preceding one.

Step 3. – We have the canonical isomorphism of $\widehat{L\mathfrak{g}}$ -modules

$$H^0(\mathcal{Q}_G^{par}(\underline{p}, \underline{P}), \pi^*(\mathcal{L}(\ell, \underline{m}))) \xrightarrow{\sim} H^0(\mathcal{Q}_G, \mathcal{O}_{\mathcal{Q}_G}(\ell)) \otimes \left(\otimes_{i=1}^n H^0(G/P_i, \mathcal{L}_{-\lambda_i}) \right)$$

To see this apply the Kunneth formula to the restriction of $\mathcal{L}(\ell, \underline{m})$ to the projective varieties $\mathcal{Q}_G^{par}(\underline{p}, \underline{P})^{(N)} = \mathcal{Q}_G(N) \times \prod_{i=1}^n G/P_i$, then use that inverse limits commute with tensor products by finite dimensional vector spaces.

Step 4. – We have the canonical isomorphism of \widehat{LG} -modules

$$H^0(\mathcal{Q}_G, \mathcal{O}_Q(\ell)) \otimes \left(\bigotimes_{i=1}^n H^0(G/P_i, \mathcal{L}_{-\lambda_i}) \right) \xrightarrow{\sim} \mathcal{H}(\ell)^* \otimes \left(\bigotimes_{i=1}^n L_{\lambda_i}^* \right)$$

This is Borel-Bott-Weil theory, in the version of Kumar-Mathieu ([Ku], [Ma]) for the first factor, and the standard version ⁽¹⁾ for the others.

The theorem follows from steps 1 to 4. As we know the dimensions (at least for classical G and G_2) for the conformal blocks ([F],[B], or [So3] for an overview) we get the Verlinde dimension formula for the spaces of generalized parabolic theta-functions.

9. Coarse Moduli spaces

Throughout this section G will be a semi-simple algebraic group over \mathbb{C} and X a smooth, complete and connected curve over \mathbb{C} of genus ≥ 2 . We will show how the previous results apply to the *coarse moduli spaces* of principal G -bundles.

(9.1) Recall that a G -bundle E over X is *semi-stable* (resp. *stable*) if for every parabolic subgroup P and for every reduction E_P of E to P , we have for every dominant character (with respect to some Borel $B \subset P$) χ of P the following inequality $\deg(E_P(\chi)) \leq 0$ (resp. $<$). A stable G -bundle E is called *regularly stable*, if moreover $\text{Aut}(E)/Z(G) = \{1\}$. By Ramanathan's [24] theorem, there are coarse moduli spaces M_G of semi-stable principal G -bundles of dimension $(g-1) \dim G$ with irreducible components, denoted M_G^τ in the following, parameterized by $\tau \in \pi_1(G)$. Moreover M_G is normal and the open subset $M_G^{\text{reg}} \subset M_G$ corresponding to regularly stable G -bundles is smooth.

(9.2) Denote Cl the group of Weil divisor classes. Fix $\tau \in \pi_1(G)$. We have a commutative diagram, with r_1 and r_2 the restrictions and c and c^{reg} the canonical morphisms:

$$\begin{array}{ccc} \text{Pic}(M_G^\tau) & \xrightarrow{c} & \text{Cl}(M_G^\tau) \\ r_1 \downarrow & & \downarrow r_2 \\ \text{Pic}(M_G^{\tau, \text{reg}}) & \xrightarrow{c^{\text{reg}}} & \text{Cl}(M_G^{\tau, \text{reg}}) \end{array}$$

By normality of M_G , the restriction r_1 is injective, by smoothness of $M_G^{\tau, \text{reg}}$, the canonical morphism c^{reg} is an isomorphism. The complement $M_G^{\tau, \text{reg}}$ in M_G^τ is of codimension ≥ 2 , except when X is of genus 2 and G maps onto PGL_2 : this is seen exactly as the analogous statement for Higgs bundles, which is proved in [9], Thm. II.6. In what follows we will assume that we are not in this exceptional case. In particular the restriction r_2 is an isomorphism. It follows that M_G^τ is locally factorial if and only if r_1 is surjective.

(9.3) Proof of 1.6. – Suppose G simply connected. The codimension of the closed substack $\mathcal{M}_G - \mathcal{M}_G^{ss}$ in \mathcal{M}_G is at least 2. To see this use the Harder-Narasimhan filtration in the case of G -bundles and calculate the codimension of the strata ([19], Section 3).

⁽¹⁾ In [Bott] only the case G/B (i.e. $\Sigma = \emptyset$) is considered but the generalization to arbitrary G/P_Σ is immediate (and well known).

This implies $\text{Pic}(\mathcal{M}_G) \xrightarrow{\sim} \text{Pic}(\mathcal{M}_G^{ss})$. It follows from Theorem 1.1 that $\text{Pic}(M_G)$ is infinite cyclic (note that the canonical morphism $\mathcal{M}_G^{ss} \rightarrow M_G$ induces an injection on the level of Picard groups). The smoothness of \mathcal{M}_G^{ss} implies that the restriction defines a surjection $\text{Pic}(\mathcal{M}_G^{ss}) \xrightarrow{f} \text{Pic}(\mathcal{M}_G^{reg})$. As $\text{Pic}(\mathcal{M}_G^{reg})$ contains $\text{Pic}(M_G^{reg})$ (which contains $\text{Pic}(M_G)$ by 9.2), the surjection f has to be an isomorphism. As the determinant line bundle exists on M_{SL_r} by Kempf's lemma [7], it follows that the generator is the determinant of cohomology for G of type A and C by (2.3). Moreover, M_G is locally factorial by (9.2) in this case. ■

(9.1) Consider $G = SO_r$ with its standard (orthogonal) representation and suppose that $r \geq 7$. The moduli space $\text{Pic}(M_{SO_r})$ is the good quotient of a parameter scheme \mathcal{R} by $GL(H)$ with $H = \mathbb{C}^{rN}$ (cf. [27]). The scheme \mathcal{R} parameterizes equivalent (with the obvious equivalence relation) triples $([F, \sigma, \alpha])$, where (F, σ) is a semi-stable SO_r -bundle and $\alpha : H^0(X, F(N)) \xrightarrow{\sim} H$. Choose a theta-characteristic κ on X . Then on \mathcal{R} there is the $GL(H)$ -linearized pfaffian of cohomology line bundle \mathcal{P}_κ deduced from the universal family over $\mathcal{R} \times X$.

PROPOSITION 9.5. – *Suppose $r \geq 7$. The line bundle \mathcal{P}_κ descends to $M_{SO_r}^{reg}(0)$ but not to $M_{SO_r}^0$. In particular, $M_{SO_r}^0$ is not locally factorial.*

Proof. – We use Kempf's lemma. If r is even, the stabilizer at a point $q = [F, \sigma, \alpha] \in \mathcal{R}^{reg}$ is ± 1 ; if r is odd, the stabilizer is reduced to 1. In the former case, by definition of the pfaffian of cohomology, using that its formation commutes with base change, the action of ± 1 at $\mathcal{P}_\kappa(q)$ is given by $g \mapsto g^{h^1(F \otimes \kappa)}$, so the action is trivial, as $h^1(F \otimes \kappa)$ is even by (7.10.1). We see that \mathcal{P}_κ descends to $M_{SO_r}^{reg}(0)$ in both cases. Let (F_1, σ_1) be a regularly stable odd SO_4 -bundle, and (F_2, σ_2) be a regularly stable odd SO_{r-4} -bundle. If $r = 8$, suppose that (F_1, σ_1) and (F_2, σ_2) are not isomorphic. Then the orthogonal sum $(F, \tau) = (F_1 \oplus F_2, \sigma_1 \oplus \sigma_2)$ is even. Let $[F, \tau, \alpha] \in \mathcal{R}$ be a point corresponding to (F, τ) . Again, by definition of the pfaffian of cohomology, using that its formation commutes with base change, we see that the action of the stabilizer $\{\pm 1\} \times \{\pm 1\}$ is

$$(g_1, g_2) \mapsto g_1^{h^1(F_1 \otimes \kappa)} g_2^{h^1(F_2 \otimes \kappa)}.$$

But the element $(-1, 1)$ acts nontrivially, so \mathcal{P}_κ does not descend to $M_{SO_r}^{reg}(0)$. ■

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