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The cup product of Hilbert schemes for K3 surfaces

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Abstract. To any graded Frobenius algebra A we associate a sequence of graded Frobenius algebras $A^{[n]}$ so that there is canonical isomorphism of rings $(H^*(X;\mathbb{Q})[2])^{[n]} \cong H^*(X^{[n]};\mathbb{Q})[2n]$ for the Hilbert scheme $X^{[n]}$ of generalised n-tuples of any smooth projective surface X with numerically trivial canonical bundle.

1. Introduction

Let X be a smooth projective surface and let $X^{[n]}$ denote the Hilbert scheme of generalized n-tuples for each non-negative integer n. The topological or geometric data of $X^{[n]}$ can be expressed in terms of the corresponding data of the surface itself intricately interwoven with the combinatorics of the symmetric group. Instances of this principle are Göttsche's formula for the Betti numbers and Nakajima's and Grojnowski's lift of this formula to the level of vector spaces with non-degenerate pairings:

$$\bigoplus_{n\geq 0} H^*(X^{[n]};\mathbb{Q}) \xrightarrow{\cong} S^*(t^{-1}\mathbb{Q}[t^{-1}]\otimes H^*(X;\mathbb{Q})).$$

This isomorphism allows one to reconstruct the cup-product pairing on $H^*(X^{[n]};\mathbb{Q})$, but not the cup product itself. The cup product structure is the topic of this paper. A complete description of the product in the framework of vertex algebra calculus was given in [14] for surfaces whose cohomology is generated by classes that are algebraic, and by Li, Qin and Wang [16] for arbitrary projective surfaces. However, these descriptions are rather implicit

Mathematics Subject Classification (1991): 14C05, 14C15, 20B30, 17B68, 17B69, 20C05 in terms of multiplication operators or explicit generators with implicitly given relations.

Our intention here is to construct a sequence of endofunctors $A \mapsto A^{[n]}$ on the category of Frobenius algebras, which when applied to the cohomology ring of a projective surface yield the cohomology rings of its Hilbert schemes. More precisely, our main result is as follows.

Theorem 1.1. — *Let X be a smooth projective surface with numerically trivial canonical divisor. Then there is a canonical isomorphism of graded rings*

$$(H^*(X;\mathbb{Q})[2])^{[n]} \xrightarrow{\cong} H^*(X^{[n]};\mathbb{Q})[2n].$$

Recall that the smooth projective surfaces with numerically trivial canonical divisor are $K3$, abelian, Enriques and bielliptic surfaces. The presence of a numerically non-trivial canonical divisor requires a deformation of this product due to a curious correction term in [14, Theorem 3.10.2], which we do not understand so far.

This generalises Vasserot's [21] and the authors' [15] theorem that there is a canonical ring isomorphism $\mathrm{gr}^F \mathcal{C}(S_n) \longrightarrow H^*((\mathbb{C}^2)^{[n]};\mathbb{Z})$, where $\mathcal{C}(S_n)$ is the centre of the group ring $\mathbb{Z}[S_n]$, and F is the so-called age filtration. Though the affine plane is not projective and hence not covered by the formalism we are going to describe, it has the advantage that its topology is trivial and that we may undisturbedly observe the combinatorial effects. The starting point of our work was provided by I. B. Frenkel and W. Wang [8] who observed that the convolution product is related to the differential operator ∂ of [14] (see Wang [22] for an overview).

B. Fantechi and L. Göttsche [5] have shown that our construction $A \mapsto A^{[n]}$ identifies to a slightly modified version of Chen's and Ruan's orbifold cohomology construction in the case of the n-fold symmetric product of X. Hence our main theorem implies Y. Ruan's conjecture 6.3 in [20] that the orbifold cohomology ring is isomorphic to the cohomology ring of the Hilbert scheme of n-points.

2. The algebraic model

2.1. In this section we will construct a sequence of endofunctors $A \to A^{[n]}$ in the category of graded Frobenius algebras.

For our purposes a graded Frobenius algebra of degree d is a finite dimensional graded vector space $A = \bigoplus_{i=-d}^{d} A^i$ with a graded commutative and associative multiplication $A \otimes A \to A$ of degree d and unit element 1 (necessarily of degree $-d$) together with a linear form $T : A \to \mathbb{Q}$ of degree $-d$ such that the induced symmetric bilinear form $\langle a, b\rangle := T(ab)$ is non-degenerate (and of degree 0). It follows from $1 \cdot 1 = 1$ that d must be an even number. The degree conventions are chosen in such a way that A is centred around degree 0. The degree of an element a will be denoted by $|a|$.

In the applications, A will be the shifted cohomology ring $H^*(X;\mathbb{Q})[d]$ of a compact complex manifold X of even dimension d.

The tensor product $A^{\otimes n}$ is again a graded Frobenius algebra of degree nd with product

$$(a_1 \otimes \ldots \otimes a_n) \cdot (b_1 \otimes \ldots \otimes b_n) := \varepsilon(\underline{a}, \underline{b})(a_1 b_1) \otimes \ldots \otimes (a_n b_n),$$

where $\varepsilon(\underline{a}, \underline{b})$ is the sign resulting from reordering the a's and b's. The integral is given by $T(a_1 \otimes \ldots \otimes a_n) := T(a_1) \cdot \ldots \cdot T(a_n)$.

The symmetric group S_n acts on the n-fold tensor product $A^{\otimes n}$ as

$$\pi(a_1 \otimes \ldots \otimes a_n) = \varepsilon(\pi, \underline{a}) a_{\pi^{-1}(1)} \otimes \ldots \otimes a_{\pi^{-1}(n)},$$

where $\varepsilon(\pi, \underline{a}) = (-1)^{\sum_{i<j, \pi(j)<\pi(i)} |a_i| \cdot |a_j|}$ is the sign introduced by interchanging the a_i's.

It will be useful to extend the definition of the n-fold tensor product to allow arbitrary unordered finite indexing sets. Therefore let I be a finite set with n elements and let $\{A_i\}_{i \in I}$ be a family of copies of a graded Frobenius algebra A indexed by I. Let $[n]$ denote the set $\{1, \ldots, n\}$. Then we define

$$A^{\otimes I} := \left(\bigoplus_{f : [n] \xrightarrow{\cong} I} A_{f(1)} \otimes \ldots \otimes A_{f(n)} \right) \Big/ S_n.$$

Here the direct sum inherits a Frobenius algebra structure by componentwise defined operations, the symmetric group acts on this direct sum via the induced operation on the set of bijections $[n] \xrightarrow{\cong} I$, and we take the quotient by the action of S_n. Thus for each bijection $f : [n] \to I$ there is a canonical isomorphism $A^{\otimes n} \xrightarrow{\cong} A^{\otimes I}$.

Let $n_1, \ldots, n_k$ be natural numbers and let $n := n_1 + \ldots + n_k$. Consider the ring homomorphism $\varphi_{n_\bullet, k} : A^{\otimes n} \to A^{\otimes k}$ which sends

$$a_1 \otimes \ldots \otimes a_n \mapsto (a_1 \cdots a_{n_1}) \otimes \ldots \otimes (a_{n_1 + \ldots + n_{k-1} + 1} \cdots a_n).$$

As before we want to generalise this to get a graded ring homomorphism $\varphi^* : A^{\otimes I} \to A^{\otimes J}$ for any surjective map $\varphi : I \to J$ of finite sets of cardinality $n := |I|$ and $k := |J|$. Choose a bijection $g : [k] \to J$ and let $n_i := |\varphi^{-1}(g(i))|$. Then there is a bijection $f : [n] \to I$ such that for each $i \in [k]$ one has

$$\varphi^{-1}(g(i)) = \{f(n_1 + \ldots + n_{i-1} + 1), \ldots, f(n_1 + \ldots + n_i)\}.$$

The composition

$$\varphi^* : A^{\otimes I} \xrightarrow{f^{-1}} A^{\otimes n} \xrightarrow{\varphi_{n_\bullet, k}} A^{\otimes k} \xrightarrow{g} A^{\otimes J}$$

is well-defined independently of the choices of f and g. If $\varphi : I \to J$ and $\psi : J \to K$ are surjections then $\psi^* \circ \varphi^* = (\psi \circ \varphi)^*$.

Finally, for any surjection $\varphi : I \to J$ let

$$\varphi_* : A^{\otimes J} \to A^{\otimes I}$$

be the linear map adjoint to φ^* with respect to the bilinear forms on $A^{\otimes I}$ and $A^{\otimes J}$. Then φ^* and φ_* are both homogeneous maps of degree $d(|I|-|J|)$. Whereas φ^* is a ring homomorphism, φ_* is a module homomorphism with respect to φ^*, i.e. the 'projection formula' holds:

$$\varphi_*(a \cdot \varphi^*(b)) = \varphi_*(a) \cdot b$$

for all $b \in A^{\otimes I}$ and $a \in A^{\otimes J}$.

2.2. Consider the composite map

$$A \xrightarrow{\Delta_*} A \otimes A \longrightarrow A, \tag{2.1}$$

where the second map is multiplication and Δ_* is the adjoint comultiplication. The image of 1 under the composite linear map is called the Euler class of A and is denoted by $e := e(A)$. Note that if A is connected, i.e. $\dim(A^{-d}) = 1$, then by duality we have $\dim(A^d) = 1$ as well. Hence there is a unique element vol $\in A^d$ such that $T(\mathrm{vol}) = 1$. In this case $e(A) = \chi(A)\mathrm{vol}$, where $\chi(A) := \sum_i (-1)^i \dim(A^i)$ is the Euler-Poincaré-characteristic of A.

2.3. For any numerical function $\nu : I \to \mathbb{N}_0$ let

$$e^\nu := \otimes_{i\in I} e^{\nu(i)} \in A^{\otimes I}.$$

For instance, if $\varphi : I \to J$ is a surjective map and $\nu(j) := |\varphi^{-1}(j)| - 1$ for all $j \in J$, then

$$\varphi^*\varphi_*(a) = e^\nu \cdot a \tag{2.2}$$

for all $a \in A^{\otimes J}$. More generally, let

$$\Box := \begin{array}{ccc} I & \xrightarrow{\alpha} & K \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ J & \xrightarrow{\delta} & L \end{array} \tag{2.3}$$

be a cocartesian diagram of finite sets and surjective maps. The associated diagram

$$\begin{array}{ccc} A^{\otimes I} & \xrightarrow{\alpha^*} & A^{\otimes K} \\ {\scriptstyle\beta_*}\uparrow & & \uparrow{\scriptstyle\gamma_*} \\ A^{\otimes J} & \xrightarrow{\delta^*} & A^{\otimes L} \end{array} \tag{2.4}$$

of linear maps is not commutative. The deviation from commutativity is measured by the following function: for any $\ell \in L$ let

$$\nu_\square(\ell) := 1 - |\delta^{-1}(\ell)| - |\gamma^{-1}(\ell)| + |(\delta\beta)^{-1}(\ell)|.$$

Then:

Lemma 2.4. — $\alpha^*\beta_*(a) = \gamma_*\left(e^{\nu_\square}\delta^*(a)\right)$ *for all* $a \in A^{\otimes L}$.

Proof. It suffices to consider the case $|L| = 1$. Let G denote the graph whose vertices consist of the sets J and K, and whose edges consist of all pairs $(\beta(i), i)$ and $(i, \alpha(i))$ for $i \in I$. The assumption that the diagram be cocartesian is equivalent to the connectedness of the graph G. Every loop of this graph gives rise to a contraction of type (2.1) and hence introduces a factor e. The number of such loops is $1 - (|J| + |K|) + |I|$. □

We leave the verification of the following observation to the reader (recall that multiplication is a homogeneous map of degree d).

Lemma 2.5. — *Let* $\nu : I \to \mathbb{N}_0$ *be a function on the finite set* I. *Then* $e^\nu \in A^{\otimes I}$ *has degree* $|e^\nu| = 2d \sum_{B \in I} \nu(B) - d|I|$.

2.6. For any permutation $\pi \in S_n$ we call

$$|\pi| := \min\{m \mid \exists \text{ transpositions } \tau_1, \dots, \tau_m \text{ such that } \pi = \tau_1 \cdot \dots \cdot \tau_m\}$$

the degree of π. The degree of π depends only on its cycle type and is sub-multiplicative:

$$|\pi\rho| \leq |\pi| \cdot |\rho| \qquad \forall \pi\rho \in S_n.$$

Since $\pi \mapsto \operatorname{sgn}(\pi) = (-1)^{|\pi|}$ is a homomorphism, the *degree defect*

$$a(\pi_1, \pi_2, \dots, \pi_t) := \frac{1}{2}(|\pi_1| + \dots + |\pi_t| - |\pi_1 \cdots \pi_t|)$$

is a non-negative integer.

For any subgroup $H \subset S_n$ and an H-stable subset $B \subset [n] := \{1, \dots, n\}$, $H\backslash B$ denotes the orbit space for the induced action. For instance, the degree can be expressed by $|\pi| = n - |\langle\pi\rangle\backslash[n]|$.

For $\pi, \rho \in S_n$ the *graph defect* $g(\pi, \rho) : \langle\pi, \rho\rangle\backslash[n] \longrightarrow \mathbb{Q}$ is defined by

$$g(\pi, \rho)(B) = \frac{1}{2}\left(|B| + 2 - |\langle\pi\rangle\backslash B| - |\langle\rho\rangle\backslash B| - |\langle\pi\rho\rangle\backslash B|\right).$$

Lemma 2.7. — *The graph defect* g *takes value in the non-negative integers.*

Proof. Treating each orbit separately, we may assume that $B = [n]$, i.e. that the group $\langle\pi, \rho\rangle$ acts transitively on $[n]$. We will identify the defect g as the genus of an oriented closed compact surface C so that the claim becomes obvious.

Let $\sigma := (\pi\rho)^{-1}$. Let C be constructed as follows: Take $\{1, \dots, n\}$ as the set of vertices. For all $i \in \{1, \dots, n\}$ and $g \in \{\pi, \rho, \sigma\}$ add an oriented edge $e_{i,g}$ from i to $g(i)$. For each i glue in a (black) triangle along the edges $e_{i,\sigma}$, $e_{\sigma(i),\rho}$, and $e_{\rho\sigma(i),\pi}$. Finally, for each $g \in \{\pi, \rho, \sigma\}$ and each orbit B' of g glue in a (white) $|B'|$-gon along the edges $e_{i,g}, e_{g(i),g}, \dots, e_{g^{|B'|-1}(i),g}$ for some element $i \in B'$. (If $|B'|$ is 1 or 2, then the $|B'|$-gon is rather degenerate: a disc with 1 respectively 2 edges.) Every edge bounds one white and one black polygon. The resulting CW-complex is a connected compact oriented surface C with Euler characteristic

$$\begin{aligned} 2 - 2g(C) = \chi(C) &= \#\text{vertices} - \#\text{edges} + \#\text{polygons} \\ &= n - 3n + (a + b + c + n), \end{aligned}$$

where a, b, and c are the numbers of black polygons corresponding to the orbits of π, ρ, and σ, respectively. Hence $g(C) = \frac{1}{2}(n + 2 - a - b - c) = g(\pi, \rho)$. □

As in the case of the degree defect the graph defect allows a natural multivariant extension: for $\pi_1, \dots \pi_t \in S_n$ let

$$g(\pi_1, \dots, \pi_t)(B) = \frac{1}{2}\left(|B| + 2 - \sum_j |\langle\pi_j\rangle\backslash B| - |\langle\pi_1 \cdots \pi_t\rangle\backslash B|\right),$$

for $B \in \langle\pi_1, \dots, \pi_t\rangle\backslash[n]$.

2.8. Now consider the graded vector space

$$A\{S_n\} := \bigoplus_{\pi\in S_n} A^{\otimes\langle\pi\rangle\backslash[n]} \cdot \pi. \tag{2.5}$$

Here the grading of an element $a\cdot\pi$ is $|a\cdot\pi| := |a|$. The symmetric group S_n acts on $A\{S_n\}$: the action of $\sigma \in S_n$ on $[n]$ induces a bijection

$$\sigma : \langle\pi\rangle\backslash[n] \to \langle\sigma\pi\sigma^{-1}\rangle\backslash[n], \quad x \mapsto \sigma x$$

for each π and hence an isomorphism

$$\tilde{\sigma} : A\{S_n\} \longrightarrow A\{S_n\}, \quad a\pi \mapsto \sigma^*(a)\sigma\pi\sigma^{-1}. \tag{2.6}$$

Let

$$A^{[n]} := (A\{S_n\})^{S_n}$$

be the subspace of invariants.

Example 2.9. — For $n = 3$ we get

$$A\{S_3\} = A^{\otimes 3}\mathrm{id} \oplus A^{\otimes 2}(12) \oplus A^{\otimes 2}(13) \oplus A^{\otimes 2}(23) \oplus A(123) \oplus A(132),$$

where we agree to give the orbits the lexicographical order with respect to $1 < 2 < 3$. Symmetrising, we find

$$A^{[3]} \cong S^3 A \oplus (A \otimes A) \oplus A.$$

2.10. Let $\mathcal{V}(A) := S^*(A \otimes t^{-1}\mathbb{Q}[t^{-1}])$ be the bosonic Fock space modelled on the graded vector space A. Then $\mathcal{V}(A)$ is bigraded by degree and weight, where an element $a \otimes t^{-m} \in A \otimes t^{-m}$ is given degree $|a|$ and weight m. The component of $\mathcal{V}(A)$ of constant weight n is the graded vector space

$$\mathcal{V}(A)_n \cong \bigoplus_{||\alpha||=n} \bigotimes_i S^{\alpha_i} A,$$

where $\alpha = (1^{\alpha_1}, 2^{\alpha_2} \cdots)$ runs through all partitions of n and $||\alpha|| := \sum_i i\alpha_i$.

If we think of $\mathbb{Q}$ as a Frobenius algebra of degree $d = 0$, then $\mathbb{Q}\{S_n\} = \mathbb{Q}[S_n]$ is the group ring and $\mathcal{V}(\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \dots]$ with $p_m = t^{-m}$ is the ring of symmetric functions. The following map generalizes the classical characteristic map $\mathbb{Q}[S_n] \to \mathbb{Q}[p_1, p_2, \dots]$.

Let $f : \{1, \dots, N\} \to \langle\pi\rangle\backslash[n]$ be an enumeration of the orbits of $\pi \in S_n$, and let $\ell_i := |f(i)|$ denote the length of the i-th orbit. Then define

$$\Phi' : A^{\otimes N} \longrightarrow \mathcal{V}(A),\, a_1 \otimes \cdots \otimes a_N \mapsto \frac{1}{n!}\big(a_1 \otimes t^{-\ell_1}\big) \cdots \big(a_N \otimes t^{-\ell_N}\big),$$

and let

$$\Phi : \bigoplus_{n \geq 0} A\{S_n\} \longrightarrow \mathcal{V}(A) \tag{2.7}$$

be given on the summand $A^{\otimes\langle\pi\rangle\backslash[n]}\pi$ by the composition

$$A^{\otimes\langle\pi\rangle\backslash[n]} \xrightarrow{f^{-1}} A^{\otimes N} \xrightarrow{\Phi'} \mathcal{V}(A).$$

Proposition 2.11. — *Φ induces an isomorphism of graded vector spaces*

$$A^{[n]} \longrightarrow \mathcal{V}(A)_n.$$

Proof. The map Φ is surjective and invariant under the S_n-action, so that its restriction to $A^{[n]}$ is also surjective. Moreover, an invariant vector v in $A^{[n]}$ is determined by its components v_π in $A^{\otimes\langle\pi\rangle\backslash[n]}$, where π runs through a system of representatives for all possible cycle types, i.e. all partitions of n, and each component v_n is symmetric with respect to exchanging the values corresponding to orbits of the same length, and conversely. This shows that both vector spaces have the same dimensions. □

Our next goal is to put a ring structure on $A\{S_n\}$ such that $A^{[n]}$ becomes a commutative subring.

2.12. Any inclusion $H \subset K$ of subgroups of S_n leads to a surjection $H\backslash[n] \twoheadrightarrow K\backslash[n]$ of orbit spaces and hence to maps

$$f^{H,K} : A^{\otimes H\backslash[n]} \longrightarrow A^{\otimes K\backslash[n]} \quad \text{and} \quad f_{K,H} : A^{\otimes K\backslash[n]} \longrightarrow A^{\otimes H\backslash[n]}. \tag{2.8}$$

If $H = \langle\pi\rangle$ is the cyclic subgroup generated by a permutation π, we omit the brackets $\langle - \rangle$ in the notation. For $\pi, \rho \in S_n$ define

$$\begin{gathered} m_{\pi,\rho} : A^{\otimes\langle\pi\rangle\backslash[n]} \otimes A^{\otimes\langle\rho\rangle\backslash[n]} \longrightarrow A^{\otimes\langle\pi\rho\rangle\backslash[n]} \\ m_{\pi,\rho}(a \otimes b) := f_{\langle\pi,\rho\rangle,\pi\rho}\left(f^{\pi,\langle\pi,\rho\rangle}(a) \cdot f^{\rho,\langle\pi,\rho\rangle}(b) \cdot e^{g(\pi,\rho)}\right) \end{gathered} \tag{2.9}$$

where $g(\pi, \rho) : \langle\pi, \rho\rangle\backslash[n] \to \mathbb{N}_0$ is the graph defect defined in 2.6.

Proposition 2.13. — *The product* $A\{S_n\} \times A\{S_n\} \xrightarrow{\cdot} A\{S_n\}$ *defined by*

$$a\pi \;\cdot\; b\rho := m_{\pi,\rho}(a \otimes b)\pi\rho$$

is associative, S_n*-equivariant, and homogeneous of degree* nd.

Proof. For any $\pi, \rho, \sigma \in S_n$ consider the following diagram of orbit spaces:

$$\begin{tikzcd}[column sep=small] \langle\pi\rangle\backslash[n] \arrow[dr] & \langle\rho\rangle\backslash[n] \arrow[d] & \langle\pi\rho\rangle\backslash[n] \arrow[dl] \arrow[dr] & \langle\sigma\rangle\backslash[n] \arrow[d] & \langle\pi\rho\sigma\rangle\backslash[n] \arrow[dl] \\ & \langle\pi,\rho\rangle\backslash[n] \arrow[dr] & \star & \langle\pi\rho,\sigma\rangle\backslash[n] \arrow[dl] & \\ & & \langle\pi,\rho,\sigma\rangle\backslash[n] & & \end{tikzcd}$$

The arrows of type $\searrow$ and $\downarrow$ will correspond to ring homomorphisms whereas arrows of type $\swarrow$ will contravariantly induce module homomorphisms. By definition, we have $(a\pi \cdot b\rho) \cdot c\sigma = Y\pi\rho\sigma$ with

$$Y = f_{\langle\pi\rho,\sigma\rangle,\pi\rho\sigma}\left(f^{\pi\rho,\langle\pi\rho,\sigma\rangle} f_{\langle\pi,\rho\rangle,\pi\rho}(Y') \cdot f^{\sigma,\langle\pi\rho,\sigma\rangle}(c) \cdot e^{g(\pi\rho,\sigma)}\right).$$

and

$$Y' = f^{\pi,\langle\pi,\rho\rangle}(a) \cdot f^{\rho,\langle\pi,\rho\rangle}(b) \cdot e^{g(\pi,\rho)}.$$

Lemma 2.4 applies to the central diamond $\star$ in the diagram above: let $\nu_\star : \langle\pi, \rho, \sigma\rangle\backslash[n] \to \mathbb{N}_0$ be defined by

$$\nu_\star(B) = 1 - |\langle\pi, \rho\rangle\backslash B| - |\langle\pi\rho, \sigma\rangle\backslash B| + |\langle\pi\rho\rangle\backslash B|.$$

Then by 2.4:

$$\begin{aligned} &f^{\pi\rho,\langle\pi\rho,\sigma\rangle} f_{\langle\pi,\rho\rangle,\pi\rho}(Y') \\ &\quad = f_{\langle\pi,\rho,\sigma\rangle,\langle\pi\rho,\sigma\rangle}(f^{\langle\pi,\rho\rangle,\langle\pi,\rho,\sigma\rangle}(Y') \cdot e^{\nu_\star}) \\ &\quad = f_{\langle\pi,\rho,\sigma\rangle,\langle\pi\rho,\sigma\rangle}\left(f^{\pi,\langle\pi,\rho,\sigma\rangle}(a) \cdot f^{\rho,\langle\pi,\rho,\sigma\rangle}(b) \cdot f^{\langle\pi,\rho\rangle,\langle\pi,\rho,\sigma\rangle}(e^{g(\pi,\rho)}) \cdot e^{\nu_\star}\right). \end{aligned}$$

We apply the projection formula and get

$$Y = f_{\langle\pi,\rho,\sigma\rangle,\pi\rho\sigma}\left(f^{\pi,\langle\pi,\rho,\sigma\rangle}(a) \cdot f^{\rho,\langle\pi,\rho,\sigma\rangle}(b) \cdot f^{\sigma,\langle\pi,\rho,\sigma\rangle}(c) \cdot Y''\right), \tag{2.10}$$

with

$$Y'' = f^{\langle\pi,\rho\rangle,\langle\pi,\rho,\sigma\rangle}(e^{g(\pi,\rho)}) \cdot e^{\nu_\star} \cdot f^{\langle\pi\rho,\sigma\rangle,\langle\pi,\rho,\sigma\rangle}(e^{g(\pi\rho,\sigma)}) =: e^h.$$

In this expression $h : \langle\pi, \rho, \sigma\rangle\backslash[n] \longrightarrow \mathbb{N}_0$ is the function

$$\begin{aligned}
h(B) &= \sum_{B'\in\langle\pi,\rho\rangle\backslash B} g(\pi,\rho)(B') + \nu_\star(B) + \sum_{B''\in\langle\pi\rho,\sigma\rangle\backslash B} g(\pi\rho,\sigma)(B'') \\
&= \frac{1}{2}\left(|B| + 2|\langle\pi,\rho\rangle\backslash B| - |\langle\pi\rangle\backslash B| - |\langle\rho\rangle\backslash B| - |\langle\pi\rho\rangle\backslash B|\right) \\
&\quad + (1 - |\langle\pi,\rho\rangle\backslash B| - |\langle\pi\rho,\sigma\rangle\backslash B| - |\langle\pi\rho\rangle\backslash B|) \\
&\quad + \frac{1}{2}\left(|B| + 2|\langle\pi\rho,\sigma\rangle\backslash B| - |\langle\pi\rho\rangle\backslash B| - |\langle\sigma\rangle\backslash B| - |\langle\pi\rho\sigma\rangle\backslash B|\right) \\
&= |B| + 1 - \frac{1}{2}\left(|\langle\pi\rangle\backslash B| + |\langle\rho\rangle\backslash B| + |\langle\sigma\rangle\backslash B| + |\langle\pi\rho\sigma\rangle\backslash B|\right) \\
&= g(\pi,\rho,\sigma)(B).
\end{aligned}$$

Summing up, we have

$$Y = f_{\langle\pi,\rho,\sigma\rangle,\pi\rho\sigma}\left(f^{\pi,\langle\pi,\rho,\sigma\rangle}(a) \cdot f^{\rho,\langle\pi,\rho,\sigma\rangle}(b) \cdot f^{\sigma,\langle\pi,\rho,\sigma\rangle}(c) \cdot e^{g(\pi,\rho,\sigma)}\right).$$

The same symmetric expression arises if we compute $a\chi_\pi \cdot (b\chi_\rho \cdot c\chi_\sigma)$ in the same way. This proves associativity.

The following terms contribute to the degree of $|a\pi \cdot b\rho|$: the degrees of a, b and $e^{g(\pi,\rho)}$ (given by Lemma 2.5), the degrees of $f^{\pi,\langle\pi,\rho\rangle}$, $f^{\rho,\langle\pi,\rho\rangle}$ and $f_{\langle\pi,\rho\rangle,\pi\rho}$, and the degrees of the two multiplications in the definition of $m_{\pi,\rho}$. The total balance is:

$$\begin{aligned}
&|a\pi \cdot b\rho| \\
&= |a| + |b| + |f^{\pi,\langle\pi,\rho\rangle}| + |f^{\rho,\langle\pi,\rho\rangle}| + |f_{\langle\pi,\rho\rangle,\pi\rho}| + |e^{g(\pi,\rho)}| + 2d|\langle\pi,\rho\rangle\backslash[n]| \\
&= |a| + |b| + d(|\langle\pi\rho\rangle\backslash[n]| - |\langle\pi,\rho\rangle\backslash[n]|) + d(|\langle\pi\rangle\backslash[n]| - |\langle\pi,\rho\rangle\backslash[n]|) \\
&\quad + d(|\langle\rho\rangle\backslash[n]| - |\langle\pi,\rho\rangle\backslash[n]|) + 2d\sum_{B\in\langle\pi,\rho\rangle\backslash[n]} g(\pi,\rho)(B) - d|\langle\pi,\rho\rangle\backslash[n]| \\
&\quad + 2|\langle\pi,\rho\rangle\backslash[n]| \\
&= |a\pi| + |b\rho| + nd
\end{aligned}$$

□

Proposition 2.14. — *For any two homogeneous elements $a\pi, b\rho \in A\{S_n\}$ the following (non)commutativity relation holds:*

$$a\pi \cdot b\rho = (-1)^{|a|\cdot|b|}\pi^*(b)\pi\rho\pi^{-1} \cdot a\pi = (-1)^{|a|\cdot|b|}\tilde{\pi}(b\rho) \cdot a\pi.$$

Proof. Let $\rho' := \pi\rho\pi^{-1}$. The following diagram of orbit spaces

$$\begin{array}{ccccccc}
 & & \langle\rho\rangle\backslash[n] & \xrightarrow{\pi} & \langle\rho'\rangle\backslash[n] & & \\
 & & \downarrow & & \downarrow & & \\
\langle\pi\rangle\backslash[n] & \longrightarrow & \langle\pi,\rho\rangle\backslash[n] & \xrightarrow{=} & \langle\rho',\pi\rangle\backslash[n] & \longleftarrow & \langle\pi\rho\rangle\backslash[n]
\end{array}$$

commutes. Hence $f^{\rho',\langle\pi,\rho'\rangle}(\pi^*(b)) = f^{\rho,\langle\pi,\rho\rangle}(b)$. Moreover, $g(\pi,\rho) = g(\rho',\pi)$. It follows that

$$\begin{aligned} m_{\pi,\rho}(a \otimes b) &= f_{\langle\pi,\rho\rangle,\pi\rho}(f^{\pi,\langle\pi,\rho\rangle}(a) \cdot f^{\rho,\langle\pi,\rho\rangle}(b) \cdot e^{g(\pi,\rho)}) \\ &= f_{\langle\rho',\pi\rangle,\pi\rho}(f^{\pi,\langle\pi,\rho\rangle}(a) \cdot f^{\rho',\langle\pi,\rho'\rangle}(\pi^*(b)) \cdot e^{g(\rho',\pi)}) \\ &= (-1)^{|a|\cdot|b|} f_{\langle\rho',\pi\rangle,\pi\rho}(f^{\rho',\langle\pi,\rho'\rangle}(\pi^*(b)) \cdot f^{\pi,\langle\pi,\rho\rangle}(a) \cdot e^{g(\rho',\pi)}) \\ &= (-1)^{|a|\cdot|b|} m_{\rho',\pi}(\pi^*(b) \otimes a) \end{aligned}$$ □

Proposition 2.15. — *$A^{[n]}$ is a subring of the centre of $A\{S_n\}$.*

Proof. This is a consequence of Proposition 2.14. □

Proposition 2.16. — *Let $T : A\{S_n\} \to \mathbb{Q}$ be defined by*

$$T(a\pi) := \begin{cases} T(a) & \text{if } \pi = \mathrm{id}, \\ 0 & \text{else}, \end{cases}$$

where T on the right hand side is the integral on $A^{\otimes[n]}$. The restriction of this integral to $A^{[n]}$ defines the structure of a graded Frobenius algebra of degree nd on $A^{[n]}$.

Proof. The integral induces an S_n-invariant bilinear form on $A\{S_n\}$. The only non-trivial pairings are of type

$$A^{\otimes\langle\pi\rangle\backslash[n]} \otimes A^{\otimes\langle\pi^{-1}\rangle\backslash[n]} \longrightarrow k.$$

As π and π^{-1} have the same orbit spaces, this map is the composition

$$A^{\otimes\langle\pi\rangle\backslash[n]} \otimes A^{\otimes\langle\pi\rangle\backslash[n]} \xrightarrow{\cdot} A^{\otimes\langle\pi\rangle\backslash[n]} \xrightarrow{\varphi_*} A^{\otimes n} \xrightarrow{T} k.$$

But $T \circ \varphi_* = T$, the integral on $A^{\otimes\langle\pi\rangle\backslash[n]}$. This shows that T induces a non-degenerate pairing on $A\{S_n\}$. Since the pairing is invariant, its restriction to $A^{[n]}$ is also non-degenerate. □

Example 2.17. — To illustrate the multiplication in $A\{S_n\}$ we take up the preceding example $n = 3$. One finds:

$$(\alpha \otimes \beta)(12) \cdot (\gamma \otimes \delta)(13) = \alpha\beta\gamma\delta(132),$$

$$(\alpha \otimes \beta)(12) \cdot (\gamma \otimes \delta)(12) = (-1)^{|\beta|\cdot|\gamma|}\Delta_*(\alpha\gamma) \otimes (\beta\delta)\mathrm{id},$$

$$\alpha(123) \cdot \beta(123) = (\alpha\beta e)(132),$$

$$\alpha(123) \cdot \beta(132) = \Delta_*(\alpha\beta)\mathrm{id}.$$

3. Hilbert schemes

3.1. If one applies the construction of the previous section to the cohomology ring of a smooth projective surface with trivial canonical divisor one obtains the cohomology ring of the Hilbert schemes. In order to do so we need to shift the degrees of the cohomology rings by the dimension of the corresponding manifold.

Let X be a smooth projective surface over the complex numbers. Let $X^{[n]} := \mathrm{Hilb}^n(X)$ denote n-th Hilbert scheme, i.e. the moduli space that represents the functor

$$S \mapsto \{Z \subset S \times X \text{ closed subscheme} \mid p : Z \to S \text{ flat, finite of degree } n\}.$$

Then X is again projective [12] and smooth [6] of dimension $2n$.

3.2. We recall some results on Hilbert schemes and their cohomology which will be used in the proof of theorem 1.1. We also take the opportunity to change some sign conventions from [14] to adapt our formulae to sign rules which are standard in the literature about vertex algebras.

Let $H := H^*(X; \mathbb{Q})[2]$ and $\mathbb{H}_n := H^*(X^{[n]}, \mathbb{Q})[2n]$. Then $\mathbb{H} := \bigoplus_{n\geq 0} \mathbb{H}_n$ is bigraded by the (shifted) cohomological degree and the conformal weight n. The shift has the effect of centring the middle degrees of the cohomology groups at zero. There is a distinguished element $\mathbf{1} \in H^0(X^{[0]}; \mathbb{Q}) \subset \mathbb{H}$, the vacuum.

As compact complex manifolds, X and the Hilbert schemes $X^{[n]}$ have fundamental classes in the top degree homology groups. We denote the evaluation of cohomology classes on the fundamental class by $\int_{[X]}$ and $\int_{[X^{[n]}]}$, respectively. Now give H and $\mathbb{H}_n$ the structure of graded Frobenius algebras by setting $T(a) := -\int_{[X]} a$ for $a \in H$ and $T(a) := (-1)^n \int_{[X^{[n]}]} a$ for $a \in \mathbb{H}_n$. Note that the two conventions agree on $X = X^{[1]}$.

3.3. For $n \in \mathbb{N}$, Nakajima defines incidence varieties

$$Z_n := \{(\xi, x, \xi') \mid \xi \subset \xi', |\xi'| - |\xi| = nx\}$$

in $X^{[\ell]} \times X \times X^{[\ell+n]}$, and operators $\mathfrak{p}_{-n} : H \to \mathrm{End}_{\mathbb{Q}}(\mathbb{H})$,

$$\mathfrak{p}_{-n}(\alpha)(y) := PD^{-1}\big(pr_{3*}\big(\big(pr_2^*(\alpha) \cup pr_1^*(y)\big) \cap [Z_n]\big)\big),$$

where $y \in \mathbb{H}_\ell$, and PD denotes Poincaré duality. Moreover, we define

$$\mathfrak{p}_n(\alpha) := \mathfrak{p}_{-n}(\alpha)^\dagger,$$

where for an endomorphism $f \in \mathrm{End}_{\mathbb{Q}}(\mathbb{H})$, $f^\dagger$ denotes the adjoint endomorphism with respect to the pairing on $\mathbb{H}$ given by T. Finally, to simplify statements, we let $\mathfrak{p}_0(\alpha) = 0$. The operators $\mathfrak{p}_{-n}(\alpha)$ are bihomogeneous of bidegree $(n, |\alpha|)$.

Theorem 3.4. — *For $n, m \in \mathbb{Z}$ and $a, b \in H$, the following oscillator relation holds:* $[\mathfrak{p}_n(a), \mathfrak{p}_m(b)] = n \cdot \delta_{n,-m} \cdot T(ab) \cdot \mathrm{id}_{\mathbb{H}}$.

Here and in the following all elements in H and $\mathbb{H}$ and, accordingly, endomorphisms of $\mathbb{H}$ are graded by their cohomological degree. The commutator in the theorem and those in the remainder of this paper have to be taken in the graded sense. The theorem is due to Nakajima [19] and Grojnowski [11], the coefficient n on the right hand side was determined by Ellingsrud and Strømme [4]. Combining the theorem with the calculation of the Betti numbers of $X^{[n]}$ due to Göttsche one obtains by formal arguments:

Theorem 3.5. — *There is an isomorphism of graded vector spaces*

$$\Psi : \mathcal{V}(H) \longrightarrow \mathbb{H}, \quad \left(a_1 t^{-n_1}\right) \cdots \left(a_s t^{-n_s}\right) \mapsto \mathfrak{p}_{-n_1}(a_1) \cdots \mathfrak{p}_{-n_s}(a_s)\mathbf{1}.$$

Recall that $\mathcal{V}(H) = S^*(H \otimes t^{-1}\mathbb{Q}[t^{-1}])$. Next, let $\Xi_n \subset X^{[n]} \times X$ denote the universal subscheme, and let $p : \Xi_n \to X^{[n]}$ and $q : \Xi_n \to X$ denote the two projections. Then for any element $\alpha \in H$, let

$$\alpha^{[n]} := p_*\big(ch(\mathcal{O}_{\Xi_n}) \cdot q^*(td(X) \cdot \alpha)\big).$$

We denote by $\alpha^{[\bullet]} \in \mathrm{End}_{\mathbb{Q}}(\mathbb{H})$ the linear operator which on $\mathbb{H}_n$ is multiplication by the class $\alpha^{[n]}$. Of particular importance is the class $1^{[n]} = ch(\mathcal{O}^{[n]})$, the Chern character of the tautological sheaf $\mathcal{O}^{[n]} := p_*(\mathcal{O}_{\Xi_n})$. The homogeneous component of degree 2 of the operator $1^{[\bullet]}$ will be denoted by ∂ (see [14]). The following proposition was first proved by the first named author [14, Thm 4.2] for the special case $\alpha = ch(F)$, F a locally free sheaf on X, and then extended to the general case by Li, Qin and Wang [16]:

Theorem 3.6. — *For all $\alpha, y \in H$ one has*

$$\big[\alpha^{[\bullet]}, \mathfrak{p}_{-1}(y)\big] = \exp(\mathrm{ad}(\partial))\mathfrak{p}_{-1}(\alpha \cdot y).$$

The theorem requires the computation of the (higher order) commutator of ∂ and $\mathfrak{p}_{-n}$. Let $\Delta_* : H \to H \otimes H$ denote the adjoint of the multiplication map. Then for any two integers $n, m \in \mathbb{Z}$ and an element $a \in H$ we obtain an operator $\mathfrak{p}_n\mathfrak{p}_m(\Delta_*(a))$. Define

$$L_n(a) := \frac{1}{2} \sum_{\nu \in \mathbb{Z}} : \mathfrak{p}_\nu \mathfrak{p}_{n-\nu} : \Delta_*(a),$$

where $: - :$ denotes the normal ordered product of two operators (i.e. $: \mathfrak{p}_\nu\mathfrak{p}_{n-\nu} : = \mathfrak{p}_\nu\mathfrak{p}_{n-\nu}$ if $n - \nu > 0$ and $\mathfrak{p}_{n-\nu}\mathfrak{p}_\nu$ otherwise). The definition of the operators $L_n(a)$ is a twisted version of the definition of the standard conformal structure for a free super boson [13]. These operators satisfy the relations

$$[L_n(a), L_m(b)] = (n-m)L_{n+m}(ab) + \delta_{n,-m}\frac{n^3 - n}{12}T(e_H ab),$$

where e_H is the Euler class of H, which is $-c_2(X)$ by our conventions, and

$$[L_n(a), \mathfrak{p}_m(b)] = -m\mathfrak{p}_{n+m}(ab).$$

The following theorem is the main result of [14]:

Theorem 3.7. — *For $n \in \mathbb{N}$ and $a \in H$ one has*

$$[\partial, \mathfrak{p}_{-n}(a)] = (-n)L_{-n}(a) + \binom{-n}{2}\mathfrak{p}_{-n}(Ka),$$

where $K \in H$ is the class of the canonical divisor of X.

Iterated application of the theorem leads to the identity

$$\frac{(-1)^n}{n!}\mathrm{ad}^n([\partial, \mathfrak{p}_{-1}(1)])(\mathfrak{p}_{-1}(a)) = \mathfrak{p}_{-n-1}(a),$$

from which it is clear that

$$\mathbb{H} = \partial\mathbb{H} + \mathfrak{p}_{-1}(H)\mathbb{H}. \tag{3.1}$$

This opens the path for many induction arguments on $\mathbb{H}$, since ∂ increases the cohomological degree, and $\mathfrak{p}_{-1}(H)$ the weight.

Remark 3.8. Theorems 3.5, 3.6 and 3.7 provide a description of the ring structure of $\mathbb{H}_n$. It is proved in [16] that the homogeneous components of the elements $a^{[n]}$, $a \in H$, form a set of generators. Although the relations among these generators are explicitly given in the sense that the theorems above provide implementable algorithms to compute all products etc., the problem remained to give a description of the resulting ring in terms of an explicit model with workable generators and relations. In this sense this paper is related to [16] as our paper [15] to the last section of [14].

The following proposition follows formally from Theorem 3.7:

Proposition 3.9. — *Assume that $K = 0$. Let $n = n_1 + \ldots + n_s$ and let $a_1, \ldots, a_s \in H$ be homogeneous classes. Then*

$$\begin{aligned}
&c_1(\mathcal{O}^{[n]}) \cdot \mathfrak{p}_{-n_1}(a_1)\cdots\mathfrak{p}_{-n_s}(a_s)\mathbf{1} \\
&= -\sum_{i<j} \varepsilon_{ij}\, n_i n_j \mathfrak{p}_{-n_i-n_j}(a_i a_j)\mathfrak{p}_{-n_1}(a_1)\cdots\widehat{\mathfrak{p}}_{-n_i}\cdots\widehat{\mathfrak{p}}_{-n_j}\cdots\mathfrak{p}_{-n_s}(a_s)\mathbf{1} \\
&\quad -\frac{1}{2}\sum_i \varepsilon_i \sum_{n'+n''=n_i} n_i \mathfrak{p}_{-n'}\mathfrak{p}_{-n''}\Delta_*(a_i)\mathfrak{p}_{-n_1}(a_1)\cdots\widehat{\mathfrak{p}}_{-n_i}\cdots\mathfrak{p}_{-n_s}(a_s)\mathbf{1},
\end{aligned}$$

where the ε's account for the signs which result from commuting the a_k's and $\widehat{\mathfrak{p}}$ indicates operators that are omitted.

Proof. By definition, multiplication by $c_1(\mathcal{O}^{[n]})$ is the same as applying the operator ∂. Now move the operator ∂ as far to the right as possible thus introducing commutators. The claim of the proposition then follows from the following facts:

1. $\partial\mathbf{1} = 0$,
2. $[[\partial, \mathfrak{p}_{-n}(a)], \mathfrak{p}_{-m}(b)] = -nm\mathfrak{p}_{-n-m}(ab)$,
3. $[\partial, \mathfrak{p}_{-n}(a)]\mathbf{1} = -\frac{1}{2}\sum_{n'+n''=n} n\mathfrak{p}_{-n'}\mathfrak{p}_{-n''}(\Delta_*(a))$.

Here (1) holds degree reasons, and (2) and (3) follow directly from 3.7. □

4. Proof of Theorem 1.1

4.1. We keep the notations of the previous sections. In the following X will always be a smooth projective surface with numerically trivial canonical divisor. $H = H^*(X;\mathbb{Q})[2]$ is a graded Frobenius algebra of degree $d = 2$ in the sense of the first section. There are isomorphisms of bigraded vector spaces

$$\Gamma : \bigoplus_{n\geq 0} H^{[n]} \xrightarrow{\ \Phi\ } \mathcal{V}(H) \xrightarrow{\ \Psi\ } \bigoplus_{n\geq 0} H^*(X^{[n]};\mathbb{Q})[2n] =: \mathbb{H}.$$

Here Φ is the map of Proposition 2.11 and Ψ is Nakajima's isomorphism in Theorem 3.5. The proof of Theorem 1.1 will be obtained by carefully identifying elements in and operators on these three spaces and translating bits of information about one ring into information about the other rings. By the very definition of Ψ the operator $a\otimes t^{-m}$ on $\mathcal{V}(H)$ corresponds to Nakajima's operator $\mathfrak{p}_{-m}(a)$ on $\mathbb{H}$. Analogously, there are operators $\mathfrak{r}_{-m}(a)$ on $\bigoplus_{n\geq 0} H^{[n]}$. We will only need the operator $\mathfrak{r}_{-1}(a)$ and give an ad hoc definition:

For any element $y \in H\{S_n\}$ and $a \in H$ let $y \otimes a \in H\{S_{n+1}\}$ denote the element that is obtained by adding the trivial cycle $(n+1)$ to each permutation $\pi \in S_n$ and tensoring the coefficient of π with a. Moreover, for all $n \in \mathbb{N}_0$ let $P : H\{S_n\} \to H^{[n]}$ be the symmetrisation operator

$$P(y) = \frac{1}{n!}\sum_{\sigma\in S_n} \widetilde{\sigma}(y).$$

Then

$$\mathfrak{r}_{-1}(a)(y) = (n+1)\cdot(-1)^{|a|\cdot|y|}P(y\otimes a).$$

Note that this defines in fact a linear map $\mathfrak{r}_{-1}(a) : H\{S_n\} \to H^{[n+1]}$. But for symmetric $y \in H^{[n]}$, we may simplify the symmetrisation operator:

$$(n+1)\cdot P(y\otimes a) = \sum_{i=1}^{n+1} (i\; n+1)^{\sim}(y\otimes a),$$

where $(i\, n+1)^{\sim}$ is the action of the transposition $(i\, n+1)$ as given by (2.6), and where by abuse of notation we allow $(i\, n+1)$ to denote the identity permutation if $i = n+1$.

Let $\Gamma := \Psi\Phi$. With the given definition of $\mathfrak{r}_{-1}$ one has

$$\Gamma(\mathfrak{r}_{-1}(a)(y)) = \Psi((a \otimes t^{-1})(\Phi(y))) = \mathfrak{p}_{-1}(a)(\Gamma(y)).$$

4.2. Let $\varepsilon_n := \sum_\pi \operatorname{sgn}(\pi)\pi \in H^{[n]}$ denote the alternating character. Recall that $\mathcal{O}^{[n]} := p_*\mathcal{O}_{\Xi_n}$ denotes the tautological sheaf on $X^{[n]}$.

Proposition 4.3. — $\Gamma(\varepsilon_n) = c(\mathcal{O}^{[n]})$.

Proof. Recall that the number of permutations π of a given cycle type $\lambda = (\lambda_1, \lambda_2, \dots) = (1^{\alpha_1}2^{\alpha_2}\cdots)$ is given by

$$C_\lambda = \frac{n!}{\prod_i \alpha_i! i^{\alpha_i}},$$

and that its signature is given by $\operatorname{sgn}(\pi) = (-1)^{\sum_{j=1}^{s}(\lambda_j - 1)} =: \operatorname{sgn}(\lambda)$. Therefore

$$\begin{aligned}\Gamma(\varepsilon_n) &= \frac{1}{n!}\sum_{\lambda \in \text{partitions of } n} C_\lambda \operatorname{sgn}(\lambda)\mathfrak{p}_{-\lambda_1}(1)\cdots\mathfrak{p}_{-\lambda_s}(1)\mathbf{1}\\ &= \sum_{\|\alpha\|=n}\prod_i \frac{1}{\alpha_i!}\left(\frac{(-1)^{i-1}\mathfrak{p}_{-i}(1)}{i}\right)^{\alpha_i}\mathbf{1}\\ &= \exp\left(\sum_i \frac{(-1)^{i-1}}{i}\mathfrak{p}_{-i}(1)\right)_n \mathbf{1},\end{aligned}$$

where the index n in the last line means that we take the component of weight n only. By [14, Theorem 4.6], the last expression equals the total Chern class of the tautological sheaf $\mathcal{O}^{[n]}$. □

If we pick only the component of degree 2, we get $c_1(\mathcal{O}^{[n]})$ on the right hand side and

$$\varepsilon_{n,2} := -\sum_{\text{all transpositions}} \tau$$

on the left hand side.

Proposition 4.4. — $\Gamma(\varepsilon_{n,2} \cdot y) = c_1(\mathcal{O}^{[n]}) \cdot \Gamma(y)$ *for all* $y \in H^{[n]}$.

Proof. This proposition is an adaptation of a result of Goulden [10, Proposition 3.1.] to our situation. Without loss of generality we may assume that y is of the form $y = P(a\pi)$. Moreover, $\varepsilon_{n,2} \cdot y = P(\varepsilon_{n,2} \cdot a\pi)$. Therefore, we may fix a permutation with a disjoint cycle decomposition $\pi = z_1 \cdots z_s$ with a certain ordering and assume that $y = (a_1 \otimes \cdots \otimes a_s)\pi$. (Of course, then y is no longer contained in $H^{[n]}$, but recall that Φ is defined on the larger ring $H\{S_n\}$.) Then by definition,

$$\Gamma(y) = \mathfrak{p}_{-\ell_1}(a_1)\cdots\mathfrak{p}_{-\ell_s}(a_s)\mathbf{1}.$$

Now let τ be a single transposition, say $\tau = (i\ j)$. We distinguish two cases according to whether i and j are contained in the same π-orbit or not. We analyse the affect of multiplying τ and π, and take the sum over all transpositions afterwards.

1. case: i and j are contained in different cycles of π, say

$$\pi = (i\, x_2 \cdots x_\ell)(j\, z_2 \cdots z_m)\cdots .$$

Then $\tau\pi = (i\, x_2 \cdots x_\ell\ j\, z_2 \cdots z_m)\cdots$, i.e. the multiplication with τ merges the two orbits. The genus defect $g(\tau, \pi)$ vanishes. Hence the multiplication map

$$m_{\tau,\pi}(1 \otimes -) : H^{\otimes\langle\pi\rangle\backslash[n]} \longrightarrow H^{\otimes\langle\tau\pi\rangle\backslash[n]} \tag{4.1}$$

is essentially given by multiplying the coefficients corresponding to the two orbits $B' := \{i, x_2, \cdots, x_\ell\}$ and $B'' := \{j, z_2, \cdots, z_m\}$. Assuming that i and j are contained in the $\tilde{\imath}$'s and $\tilde{\jmath}$'s orbit of π, this consideration shows

$$\Gamma(\tau \cdot y) = \varepsilon'\mathfrak{p}_{-\ell_{\tilde{\imath}}-\ell_{\tilde{\jmath}}}(a_{\tilde{\imath}}a_{\tilde{\jmath}})\mathfrak{p}_{-\ell_1}(a_1)\cdots\widehat{\mathfrak{p}}_{-\ell_{\tilde{\imath}}}\cdots\widehat{\mathfrak{p}}_{-\ell_{\tilde{\jmath}}}\cdots\mathfrak{p}_{-\ell_s}(a_s), \tag{4.2}$$

where ε' is the sign arising from the permutation of the a_k's. If τ runs through all transpositions, there are $|B'|\cdot|B''| = \ell_{\tilde{\imath}}\ell_{\tilde{\jmath}}$ possibilities to hit the orbits B' and B''. Thus the right hand side in (4.2) occurs with multiplicity $\ell_{\tilde{\imath}}\ell_{\tilde{\jmath}}$. Up to the sign, this yields the first term on the right hand side of Proposition 3.9.

2. case: i and j are contained in the same cycle of π, say

$$\pi = (i\, x_2 \cdots x_{m'}\ j\, z_2 \cdots z_{m''})\cdots .$$

Then $\tau\pi = (i\, x_2 \cdots x_{m'})(j\, z_2 \cdots z_{m''})\cdots$, i.e. the given cycle is split into two smaller cycles. Again the genus defect vanishes. The multiplication (4.1) is essentially given by the comultiplication $H \to H \otimes H$, where the two factors on the right hand side correspond to the two new orbits. Hence if the cycle $(i\, x_2 \cdots x_{m'}\ j\, z_2 \cdots z_{m''})$ is, say, z_h, then

$$\Gamma(\tau \cdot y) = \varepsilon''\mathfrak{p}_{-m'}\mathfrak{p}_{-m''}(\Delta_*(a_h))\mathfrak{p}_{-\ell_1}(a_1)\cdots\widehat{\mathfrak{p}}_{-\ell_h}\cdots\mathfrak{p}_{-\ell_s}(a_s), \tag{4.3}$$

where again ε'' is the sign arising from the permutation of the a_k's. There are precisely ℓ_h choices of (ordered!) pairs (i, j) from the cycle z_h such that the cycle splits into two cycles of lengths m' and m''. Again up to the sign, this corresponds to the second term on the right hand side of Proposition 3.9. Note that the factor $\frac{1}{2}$ arises since transpositions are unordered.

Summing up we see, that multiplication by $\varepsilon_{n,2}$ has the same effect – via Γ – as multiplication by $c_1(\mathcal{O}^{[n]})$ as described by Proposition 3.9. $\square$

4.5. It follows from [14, Theorem 4.2.] that

$$c(\mathcal{O}^{[n+1]})\mathfrak{p}_{-1}(a) - \mathfrak{p}_{-1}(a)c(\mathcal{O}^{[n]}) = [\partial, \mathfrak{p}_{-1}(a)]c(\mathcal{O}^{[n]}) \tag{4.4}$$

as operators on $\mathbb{H}_n$. We must prove that the corresponding assertion holds for the alternating character.

Proposition 4.6. — *The following identity of operators on $H^{[n]}$ holds:*

$$\varepsilon_{n+1}\mathfrak{r}_{-1}(a) - \mathfrak{r}_{-1}(a)\varepsilon_n = (\varepsilon_{n+1,2}\mathfrak{r}_{-1}(a) - \mathfrak{r}_{-1}(a)\varepsilon_{n,2})\varepsilon_n$$

Proof. The embedding $S_n \to H\{S_n\}$ preserves products provided that $|\pi\rho| = |\pi| \cdot |\rho|$. Therefore, there are identities

$$\varepsilon_{n+1,2} - \iota(\varepsilon_{n,2}) = \sum_{i=1}^{n}(i\ n+1) \quad \text{and} \quad \varepsilon_{n+1} - \iota(\varepsilon_n) = \sum_{i=1}^{n}(i\ n+1)\cdot\iota(\varepsilon_n).$$

The proposition follows from this by a simple calculation (see the proof of [15, Proposition 5.1.]. □

Proposition 4.7. — *The following identity holds for all $y \in H^{[n]}$:*

$$\Gamma(\varepsilon_n \cdot y) = c(\mathcal{O}^{[n]}) \cdot \Gamma(y).$$

Proof. The assertion follows from (3.1), (4.4) and Proposition 4.6 by induction on cohomological degree and weight. The calculation itself is identical to the proof of [15, Proposition 5.3.]. □

4.8. Let $TH^{[n]} \subset H^{[n]}$ and $T\mathbb{H}_n \subset \mathbb{H}_n$ denote the tautological rings, i.e. the subalgebras generated by the components $\varepsilon_{n,2k}$ of the alternating character and the components $c(\mathcal{O}^{[n]})_k$ of the tautological bundle, respectively. What we have proved so far can be rephrased as follows: Γ maps $TH^{[n]}$ to $T\mathbb{H}_n$ and, more precisely, $TH^{[n]} \to T\mathbb{H}_n$ is an isomorphism of rings, and $H^{[n]} \to \mathbb{H}_n$ is an isomorphism of modules over the tautological rings. It still remains to show that Γ is an isomorphism of rings. To see this we show that the ring structure of either ring is completely determined by the module structure over the tautological subring. The key point here is, that certain operators satisfy what Li, Qin and Wang [16] call the transfer property: for the classes $a^{[n]}$ it follows directly from Theorem 3.6 that

$$\left[a^{[\bullet]}, \mathfrak{p}_{-1}(b)\right] = \left[1^{[\bullet]}, \mathfrak{p}_{-1}(ab)\right] \tag{4.5}$$

We proceed in three steps. In Sect. 4.9 we identify the elements $\Gamma^{-1}(a^{[n]}) \in H^{[n]}$; in Sect. 4.11 we show that the elements $\Gamma^{-1}(a^{[n]})$ have the transfer property; and, finally, in Sect. 4.13 we use the transfer property to complete the proof by induction.

4.9. For $a \in H$ consider the sum $\sum_{n\geq 0} a^{[n]}$ (in the formal completion of $\mathbb{H}$ with respect to the filtration by conformal weight). We put $\mathfrak{p} := \mathfrak{p}_{-1}(1)$ for convenience and get

$$\begin{aligned}
\sum_{n\geq 0} a^{[n]} &= a^{[\bullet]} \exp(\mathfrak{p})\mathbf{1} \\
&= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n-1} \mathfrak{p}^{n-k-1}[a^{[\bullet]}, \mathfrak{p}]\mathfrak{p}^k \mathbf{1}, \\
&\quad \text{since } a^{[\bullet]}\mathbf{1} = 0, \\
&= \sum_{n\geq 0} \sum_{k=0}^{n-1} \frac{1}{n!} \mathfrak{p}^{n-k-1} \sum_{s=0}^{k} \binom{k}{s} \mathfrak{p}^{k-s}(-\mathrm{ad}\mathfrak{p})^s([a^{\bullet]}, \mathfrak{p}])\mathbf{1} \\
&= \sum_{m=0}^{\infty} \frac{\mathfrak{p}^m}{m!} \cdot \sum_{s=1}^{\infty} \frac{1}{s!} (-\mathrm{ad}\mathfrak{p})^{s-1}([a^{[\bullet]}, \mathfrak{p}])\mathbf{1}.
\end{aligned}$$

Inserting

$$[a^{[\bullet]}, \mathfrak{p}] = \exp(\mathrm{ad}\partial)\mathfrak{p}_{-1}(a)$$

we get

$$\sum_{n\geq 0} a^{[n]} = \exp(\mathfrak{p}) \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-\mathrm{ad}\mathfrak{p})^{s-1}}{s!} \frac{(\mathrm{ad}\partial)^k}{k!} (\mathfrak{p}_{-1}(a))\mathbf{1}. \tag{4.6}$$

Let α be a partition of length $|\alpha|$ and let $\Delta_* : H \to H^{\otimes|\alpha|}$ be the diagonal map as before. For $a \in H$ we define the operator

$$\mathfrak{p}_{-\alpha}(a) := \prod_{i\geq 1} (\mathfrak{p}_{-i})^{\alpha_i}(\Delta_*(a)).$$

Proposition 4.10. — *For each partition α there are rational numbers c'_α, c''_α such that for $c_\alpha := c'_\alpha + c''_\alpha e \in H$ the following identity holds:*

$$\sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-\mathrm{ad}\mathfrak{p})^{s-1}}{s!} \frac{(\mathrm{ad}\partial)^k}{k!} (\mathfrak{p}_{-1}(a))\mathbf{1} = \sum_{\alpha} \frac{1}{\|\alpha\|!} \mathfrak{p}_{-\alpha}(ac_\alpha)\mathbf{1}. \tag{4.7}$$

Pondering the left hand side the accuracy of this proposition becomes manifest, even more so as we are only claiming something about the structure of the right hand side and nothing about the coefficients themselves. However, we find it painful to formally prove the assertion and know no simpler way than doing some vertex algebra calculus. We postpone the proof to Subsect. 4.14.

By (4.6) and (4.7) we have

$$a^{[n]} = \sum_{||\alpha||\leq n} \frac{\mathfrak{p}^{n-\|\alpha\|}}{(n - \|\alpha\|)!} \frac{\mathfrak{p}_{-\alpha}(c_\alpha a)}{\|\alpha\|!} \mathbf{1}. \tag{4.8}$$

4.11. For any partition $\alpha = (1^{\alpha_1}2^{\alpha_2}\cdots)$ of $m = \|\alpha\| = \sum_i i\alpha_i$ of length $|\alpha| = \sum_i \alpha_i$ choose a permutation $\pi \in S_m$ of cycle type α. As before, let $\Delta_* : H \to H^{\otimes|\alpha|}$ be the map adjoint to the multiplication. For all $n \in \mathbb{N}_0$ let

$$B_\alpha(u)_n := \binom{n}{m} P(\Delta_*(u)\pi \otimes 1^{\otimes n-m}) \in H^{[n]},$$

where P is the symmetrisation operator. One checks that

$$\Gamma(B_\alpha(u)_n) = \frac{\mathfrak{p}^{n-\|\alpha\|}}{(n-\|\alpha\|)!}\cdot\frac{\mathfrak{p}_{-\alpha}(u)}{\|\alpha\|!}\mathbf{1}. \tag{4.9}$$

Furthermore, let $B_\alpha(u)_\bullet$ denote the operator on $\bigoplus_{n\geq 0} H^{[n]}$ which is multiplication by the elements $B_\alpha(u)_n$.

Proposition 4.12. — *The elements $B_\alpha(u)_n$ satisfy*

$$[B_\alpha(u)_\bullet, \mathfrak{r}_{-1}(b)] = [B_\alpha(1)_\bullet, \mathfrak{r}_{-1}(ub)].$$

In particular, $\Gamma^{-1}(a^{[n]})$ has the transfer property

$$\left[\Gamma^{-1}(a^{[\bullet]}), \mathfrak{r}_{-1}(b)\right] = \left[\Gamma^{-1}(1^{[\bullet]}), \mathfrak{r}_{-1}(ab)\right].$$

Proof. Let $y \in H^{[n]}$ be given. We compute the terms $B_\alpha(u)_{n+1}\cdot\mathfrak{r}_{-1}(b)(y)$ and $\mathfrak{r}_{-1}(ub)(B_\alpha(1)_n\cdot y)$:

$$\begin{aligned}
&(-1)^{|y|\cdot|b|}B_\alpha(u)_{n+1}\cdot\mathfrak{r}_{-1}(b)(y)\\
&\qquad= \binom{n+1}{m} P\big(\big(\Delta_*(u)\pi\otimes 1^{\otimes n+1-m}\big)\cdot(n+1)P(y\otimes b)\big)\\
&\qquad= \binom{n+1}{m} P\left(\left(\Delta_*(u)\pi\otimes 1^{\otimes n+1-m}\right)\cdot\textstyle\sum_{i=1}^{n+1}(i\,n+1)^\sim(y\otimes b)\right)\\
&\qquad= \binom{n+1}{m}\textstyle\sum_{i=1}^{n+1} P\left(((i\,n+1)^\sim(\Delta_*(u)\pi\otimes 1^{\otimes n+1-m}))\cdot(y\otimes b)\right)
\end{aligned}$$

If in this sum the index i takes values $m+1,\ldots,n+1$, then the transposition $(i\,n+1)$ permutes the final 1's in $\Delta_*(u)\otimes 1^{\otimes n+1-m}$ and thus has no effect. This part of the sum therefore equals

$$\begin{aligned}
&= \binom{n+1}{m}(n+1-m)P((\Delta_*(u)\pi\otimes 1^{\otimes n-m}\cdot y)\otimes b)\\
&= \binom{n+1}{m}\tfrac{n+1-m}{n+1}\,\mathfrak{r}_{-1}(b)(\Delta_*(u)\otimes 1^{\otimes n-m}\cdot y)\\
&= (-1)^{(|y|+|u|)|b|}\mathfrak{r}_{-1}(b)(B_\alpha(u)_n\cdot y).
\end{aligned}$$

This is the second half of the commutator. Adding up, we find

$$[B_\alpha(u)_\bullet, \mathfrak{r}_{-1}(b)] \\ = (-1)^{|b|\cdot|u|}\binom{n+1}{m} \sum_{i=1}^m P\left((i\,n+1)^\sim(\Delta_*(u)\otimes 1^{n+1-m}\cdot(y\otimes b)\right).$$

There is no need to evaluate the right hand side. It suffices to note that conjugation by $(i\,n+1)$ has the effect of making the point $(n+1)\in[n+1]$ part of the orbits which support the diagonally embedded class $\Delta_*(u)$. By the definition of the multiplication in $A\{S_{n+1}\}$, this leads to a contraction of u and b. But the result of the multiplication does not change if we replace the pair (u, b) by $(1, ub)$ or $(ub, 1)$. This is all we need. The second claim follows from the first when combined with (4.8) and (4.9). □

4.13. Now we are ready to finish the proof of Theorem 1.1. We must prove that for all $a\in H$ and homogeneous $y\in H^{[n]}$ one has

$$\Gamma(\Gamma^{-1}(a^{[n]})\cdot y) = a^{[n]}\cdot\Gamma(y). \tag{4.10}$$

Since $1^{[n]} = ch(\mathcal{O}^{[n]})$ is contained in the tautological ring, we already know that

$$\Gamma(\Gamma^{-1}(1^{[n]})\cdot y) = 1^{[n]}\cdot\Gamma(y). \tag{4.11}$$

We argue by induction on conformal weight and cohomological degree and assume that (4.10) is true for all conformal weights $< n$ and all degrees $< |y|$. We know that

$$\mathbb{H}_n = \partial\mathbb{H}_n + \mathfrak{p}_{-1}(1)\mathbb{H}_{n-1} \quad\text{and}\quad H^{[n]} = \varepsilon_{n,2}\cdot H^{[n]} + \mathfrak{r}_{-1}(1)H^{[n-1]}.$$

It therefore suffices to consider the following two cases:

Case 1: $y = \varepsilon_{n,s}\cdot z$. Then

$$\begin{aligned}\Gamma(\Gamma^{-1}(a^{[n]})\cdot y) &= \Gamma(\Gamma^{-1}(a^{[n]})\cdot\varepsilon_{n,2}\cdot z) = \Gamma(\varepsilon_{n,s}\cdot\Gamma^{-1}(a^{[n]})\cdot z)\\ &= \Gamma(\varepsilon_{n,2})\cdot\Gamma(\Gamma^{-1}(a^{[n]})\cdot z) = \partial\cdot a^{[n]}\Gamma(z)\\ &\qquad\text{by induction,}\\ &= a^{[n]}\partial\Gamma(z) = a^{[n]}\cdot\Gamma(\varepsilon_{n,2}\cdot z) = a^{[n]}\Gamma(y).\end{aligned}$$

Case 2: $y = \mathfrak{r}_{-1}(1)z$. Here we make use of the transfer property (Proposition 4.12) which explicitly says:

$$\begin{aligned}&\Gamma^{-1}(a^{[n]})\mathfrak{r}_{-1}(1) - \mathfrak{r}_{-1}(1)\Gamma^{-1}(a^{[n-1]})\\ &\qquad = \Gamma^{-1}(1^{[n]})\mathfrak{r}_{-1}(a) - \mathfrak{r}_{-1}(a)\Gamma^{-1}(1^{[n-1]}).\end{aligned} \tag{4.12}$$

Then

$$\begin{aligned}
\Gamma(\Gamma^{-1}(a^{[n]})\cdot y) &= \Gamma(\Gamma^{-1}(a^{[n]})\cdot \mathfrak{r}_{-1}(1)z)\\
&= \Gamma(\mathfrak{r}_{-1}(1)\Gamma^{-1}(a^{[n-1]})\cdot z)\\
&\quad +\Gamma(\Gamma^{-1}(1^{[n]})\mathfrak{r}_{-1}(a)z) - \Gamma(\mathfrak{r}_{-1}(a)\Gamma^{-1}(1^{[n-1]})\cdot z)\\
&\quad \text{because of (4.12)},\\
&= \mathfrak{p}_{-1}(1)\Gamma(\Gamma^{-1}(a^{[n-1]})z)\\
&\quad +1^{[n]}\cdot\Gamma(\mathfrak{r}_{-1}(a)\cdot z) - \mathfrak{p}_{-1}(a)1^{[n-1]}\cdot\Gamma(z)\\
&\quad \text{because (4.10) holds true for } 1^{[\nu]}\in T\mathbb{H},\\
&= \mathfrak{p}_{-1}(1)a^{[n-1]}\Gamma(z)\\
&\quad +1^{[n]}\mathfrak{p}_{-1}(a)1^{[n-1]}\Gamma(z) - \mathfrak{p}_{-1}(a)1^{[n-1]}\cdot\Gamma(z)\\
&\quad \text{by induction},\\
&= a^{[n]}\mathfrak{p}_{-1}(1)\Gamma(z)\\
&\quad \text{because of the transfer property},\\
&= a^{[n]}\Gamma(\mathfrak{r}_{-1}(1)z) = a^{[n]}\Gamma(y).
\end{aligned}$$

This finishes the proof of the main theorem (up to the proof of Proposition 4.10 in the next section).

4.14. *Proof of Proposition* 4.10. Let $\mathfrak{glf}(\mathbb{H}) \subset \mathrm{End}(\mathbb{H})[|z, z^{-1}|]$ denote the general linear field algebra of $\mathbb{H}$. Our basic fields are

$$\varphi(a)(z) := \sum_{n\in\mathbb{Z}} \mathfrak{p}_n(a)z^{-n-1}$$

for $a \in H$ and their derivatives

$$\varphi(a)^{(k)}(z) := \left(\frac{\partial}{\partial z}\right)^k \varphi(a)(z).$$

More generally, for any partition $\beta = 1^{\beta_1}2^{\beta_2}\cdots$ let

$$\varphi_\beta(a)(z) := \quad :\prod_i \left(\frac{\varphi^{(i-1)}}{i!}\right)^{\beta_i} \Delta_*(a) :,$$

where as before $\Delta_* : H \to H^{\otimes\|\beta\|}$ is the map adjoint to multiplication.

Then $\varphi_\beta(a)(z)$ is a field of conformal weight $\|\beta\|$. We recover the operators $\mathfrak{p}(a)$ and ∂ as Fourier modes of $\varphi(a)(z)$ and $\frac{1}{3!}\varphi_{1^3}(1)(z)$:

$$\varphi(a)(z) = \ldots + \mathfrak{p}(a)\cdot z^0 + \ldots \quad \text{and} \quad \frac{1}{3!}\varphi_{1^3}(1)(z) = \ldots + (-\partial)\cdot z^{-3} + \ldots$$

(For the latter fact see [7]). The Wick Theorem (cf. [13][Thm. 3.3.]) applies to these fields and yields the following OPE:

$$\begin{aligned}\varphi(1)(z)\cdot\varphi_\beta(b)(w) &\sim \sum_{j\geq 1}\beta_j\varphi_{\beta-j^1}(b)(w)\frac{1}{j!}\frac{\partial^{j-1}}{\partial w^{j-1}}\frac{1}{(z-w)^2}\\ &\sim \sum_{j\geq 1}\frac{\beta_j\varphi_{\beta-j^1}(b)(w)}{(z-w)^{j+1}},\end{aligned}$$

which implies the commutator relation (cf. [13][Thm. 2.3.])

$$-\mathrm{ad}\,\mathfrak{p}\big(\varphi_\beta(b)(w)\big) = \sum_{j\geq 1}(-w)^{-j-1}\beta_j\varphi_{\beta-j^1}(b)(w).$$

Here and in the following $\beta - j^1$ denotes the partition that equals β with the number of j's decreased by one, similarly for $\beta + j^1 - k^2$ etc.

In a similar way, we compute the OPE for the fields $\frac{1}{3!}\varphi_{1^3}(1)(z)$ and $\varphi_\beta(b)(z)$. In this case, the Wick Theorem gives several terms on the right hand side depending on whether we contract 1, 2 or 3 factors. Observe, however, that contracting N factors introduces the power e^{N-1} of the Euler class. As e^2 is zero, only the following terms are left:

$$\begin{aligned}\frac{1}{3!}\varphi_{1^3}(1)(z)\cdot\varphi_\beta(b)(w) \sim \frac{1}{2}\sum_{j\geq 1}\beta_j :\varphi(z)^2\varphi_{\beta-j^1}(w): (b)\frac{1}{(z-w)^{j+1}}\\ +\sum_{1\leq k<\ell}\beta_k\beta_\ell :\varphi(z)\varphi_{\beta-k^1-\ell^1}(w): (be)\frac{1}{(z-w)^{k+\ell+2}}\\ +\sum_{1\leq k}\binom{\beta_k}{2} :\varphi(z)\varphi_{\beta-k^2}(w): (be)\frac{1}{(z-w)^{2k+2}}\end{aligned}$$

which by Taylor expansion yields:

$$\begin{aligned}&\sim \frac{1}{2}\sum_{j\geq 1}\sum_{k,\ell\geq 1}\beta_j k\ell :\varphi_{\beta-j^1+k^1+\ell^1}(b)(w): \frac{1}{(z-w)^{j-k-\ell+3}}\\ &+\sum_{1\leq j}\sum_{1\leq k<\ell}\beta_k\beta_\ell j :\varphi_{\beta+j^1-k^1-\ell^1}(be)(w): \frac{1}{(z-w)^{k+\ell-j+3}}\\ &+\sum_{1\leq j}\sum_{1\leq k}\binom{\beta_k}{2}j :\varphi_{\beta+j^1-k^2}(be)(w): \frac{1}{(z-w)^{2k-j+3}}.\end{aligned}$$

As before we pass to the commutator relation for the Fourier mode ∂ of $\frac{1}{3!}\varphi_{1^3}(1)(z)$ and get:

$$(-\mathrm{ad}\,\partial)\big(\varphi_\beta(b)(w)\big) = \sum_{m=0}^{2}\binom{2}{m}w^m\Big\{\frac{1}{2}\sum_{\substack{j=k+\ell-m\\1\le k,\ell}}\beta_j k\ell\varphi_{\beta-j^1+k^1+\ell^1}(b)(w)$$
$$+\sum_{\substack{j=k+\ell+m\\1\le k<\ell}}\beta_k\beta_\ell j\varphi_{\beta+j^1-k^1-\ell^1}(be)(w)$$
$$+\sum_{\substack{j=2k+m\\1\le k}}\binom{\beta_k}{2}j\varphi_{\beta+j^1-k^2}(be)(w)\Big\}$$

These commutator relations can be expressed more elegantly as follows: The map

$$\begin{array}{rcl}\mu : H[z,z^{-1}][t_1,t_2,\ldots] & \xrightarrow{\cong} & \mathbb{Q}[z,z^{-1}]\langle\varphi_\beta(a)(z)\rangle_{a,\beta}\subset\mathfrak{glf}(\mathbb{H})\\ at_1^{\beta_1}t_2^{\beta_2}\cdots t_s^{\beta_s} & \mapsto & \varphi_\beta(a)(z)z^{\|\beta\|}\end{array}$$

is a $\mathbb{Q}[z,z^{-1}]$-linear isomorphism onto the submodule generated by the fields $\varphi_\beta(a)(z)$. With respect to this identification, the operators $\mathrm{ad}\,\mathfrak{p}$ and $\mathrm{ad}\,\partial$ as calculated above can be written as

$$-\mathrm{ad}\,\mathfrak{p} = z^{-1}D_1 \quad\text{and}\quad \mathrm{ad}\,\partial = -D_2$$

with differential operators

$$D_1 := \sum_{j\ge 1}(-1)^{j-1}\frac{\partial}{\partial t_j}$$

and

$$D_2 := \frac{1}{2}\sum_{k+\ell\ge j}\binom{2}{k+\ell-j}k\ell t_k t_\ell\frac{\partial}{\partial t_j} + \frac{e}{2}\sum_{j\ge k+\ell}\binom{2}{j-k-\ell}jt_j\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_\ell}.$$

To compute the left hand side of (4.7) we calculate in $H[z,z^{-1}][t_1,t_2,\ldots]$. Recall that $at_1 \mapsto \varphi(a)(z)\cdot z$ and that $\mathfrak{p}_{-1}(a)$ is the coefficient of z in this field. Then

$$f := \exp(-D_2)(t_1)$$

is a power series in $\mathbb{Q}[e][|t_1,t_2,\ldots|]$ and we need the coefficient of z in the field corresponding to $a\cdot\sum_{s\ge 1}\frac{1}{s!}z^{1-s}D_1^{s-1}f$ when applied to the vacuum, or equivalently, the term

$$\sum_{s\ge 1}\mathrm{Coeff}\Big(z^s,\mu\Big(\frac{a}{s!}D_1^{s-1}f\Big)\mathbf{1}\Big).$$

To apply the field φ or normal ordered products of its derivatives to the vacuum simply requires to throw away all Fourier modes $\mathfrak{p}_m$, $m \geq 0$, i.e. we evaluate

$$\tilde{\mu} : t_i \mapsto \sum_{n>0} \frac{\mathfrak{p}_{-n}}{n} \binom{n}{i} z^n \in \operatorname{End}(\mathbb{Q}[\mathfrak{p}_{-1}, \mathfrak{p}_{-2}, \dots])[z].$$

Now we are done: For

$$\sum_{s\geq 1} \frac{1}{s!} \operatorname{Coeff}\big(z^s, \tilde{\mu}\big(D_1^{s-1} f\big)\big) \in \mathbb{Q}[e][|\mathfrak{p}_{-1}, \mathfrak{p}_{-2}, \dots |]$$

is a power series in $\mathfrak{p}_{-\nu}$'s and hence can be expressed as

$$\sum_{s\geq 1} \frac{1}{s!} \operatorname{Coeff}\big(z^s, \tilde{\mu}\big(D_1^{s-1} f\big)\big) = \sum_{\alpha=(1^{\alpha_1}\dots r^{\alpha_r})} \frac{c_\alpha}{\|\alpha\|!} \mathfrak{p}_{-1}^{\alpha_1} \mathfrak{p}_{-2}^{\alpha_2} \cdots \mathfrak{p}_{-r}^{\alpha_r}$$

for suitable coefficients $c_\alpha \in \mathbb{Q} + \mathbb{Q}e \subset H$, so that in total

$$\sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-\operatorname{ad} \mathfrak{p})^{s-1}}{s!} \frac{(\operatorname{ad} \partial)^k}{k!} (\mathfrak{p}_{-1}(a))\mathbf{1} = \sum_{\alpha} \frac{1}{\|\alpha\|!} \mathfrak{p}_{-\alpha}(c_\alpha \cdot a) \cdot \mathbf{1}.$$

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