

THETA FUNCTIONS ON THE MODULI SPACE OF PARABOLIC BUNDLES

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1. Introduction

Let X be a smooth, connected projective curve of genus $g \geq 2$ over the field of complex numbers and I a finite subset of points of X . Let \mathcal{M}^{par} denote the moduli space of semistable parabolic vector bundles of rank r , trivial determinant and fixed parabolic structure at I . There is a natural ample line bundle \mathcal{L}^{par} on \mathcal{M}^{par} , which is the analogue of the determinant bundle \mathcal{D} on the moduli space \mathcal{M} , of vector bundles on X of fixed rank and determinant ([N-R] theorem 1, for rank 2, [Pa1] theorem 3.3, for any rank). In this paper we determine an integer ℓ_0 such that, if $\ell \geq \ell_0$, then $\mathcal{L}^{par \otimes \ell}$ is globally generated.

The analogue problem in the classical case has been studied by Faltings, Le Potier and Popa. For vector bundles, there are natural global sections (of each power h) of the determinant bundle, that are called theta functions (of order h). Faltings has shown that such sections do generate $\mathcal{D}^{\otimes h}$, for $h \gg 0$ [F], and an effective bound on h has been given by Le Potier [LP]. Recently Popa has produced a considerably better bound, in the sense that it does not depend on the genus g of the curve [Po].

The parabolic case in rank 2 has been studied by Pauly [Pa2]. He produces sections of the parabolic determinant \mathcal{L}^{par} on the moduli space of semistable parabolic bundles of rank 2 and trivial determinant. They generalize the sections of type theta of the determinant line bundle. Moreover, under the assumption that the parabolic subset I has small and even cardinality, he proves that these sections generate the line bundle \mathcal{L}^{par} .

Our main result can be stated as follows.

Theorem 1.1. *Let ℓ be an integer such that*

$$\ell \geq \left\lceil \frac{r^2}{4} \right\rceil,$$

and suppose it is $I \neq \emptyset$. Then the linear system $|\mathcal{L}^{par \otimes \ell}|$ is base point free.

We are actually going to prove that, for ℓ given by this bound, there exist global sections, the parabolic analogues of theta functions, generating $\mathcal{L}^{par \otimes \ell}$. These sections are obtained generalizing Pauly's method [Pa2] and will be called parabolic theta functions. They are associated with parabolic bundles whose rank, degree

and parabolic invariants depend on the invariants of the bundles parametrized by \mathcal{M}^{par} and on the order ℓ . Let \mathcal{M}'_ℓ denote the moduli space of semistable parabolic bundles with which we associate parabolic theta functions of order ℓ .

The idea of the proof is to show that, under this assumption on ℓ , for each point x of the moduli space \mathcal{M}^{par} , the dimension of the subscheme of points of \mathcal{M}'_ℓ , whose associated parabolic theta function vanishes at x , is strictly smaller than $\dim(\mathcal{M}'_\ell)$.

Our method of proof is inspired by Popa's beautiful ideas [Po]. An essential step in his proof is the estimate of the dimension of Grothendieck's Quot scheme (see also [Po-Ro]). This allows him to estimate the dimension of the family of bundles, that are images of a morphism from a fixed vector bundle E .

In order to treat the parabolic case, given a point F_* of \mathcal{M}'_ℓ , we first show how to identify the zeroes of the associated section with the points E_* of \mathcal{M}^{par} , admitting a nonzero parabolic morphism to F_* . Particular care has then to be taken in order to understand the family of such morphisms for which we construct a scheme that can be seen as a parabolic analogue of Grothendieck's Quot scheme. An estimate of the dimension of this scheme is given by a formula (see theorem 3.6), which extends the result of Popa and Roth [Po-Ro] to the parabolic case. The computation can then be worked out, by applying Lange's results on families of extensions to the parabolic context.

Let $\mathcal{QPar}_{X,I,\underline{n}}$ be the (algebraic) stack of quasi-parabolic vector bundles with trivial determinant and fixed quasi-parabolic structure. It is well known that the choice of a system $\underline{\alpha}$ of Seshadri parabolic weights defines a notion of $\underline{\alpha}$ -semistability for such bundles. Actually by the classification of line bundles on $\mathcal{QPar}_{X,I,\underline{n}}$ of Pauly [Pa1] and Laszlo and Sorger [La-So] the choice of $\underline{\alpha}$ also defines a line bundle $L_{\underline{\alpha}}$ on this stack and all "ample" line bundles arise this way. The above theorem may be applied to show that for ℓ sufficiently large, the base locus of the linear system $|L_{\underline{\alpha}}^{\otimes \ell}|$ on $\mathcal{QPar}_{X,I,\underline{n}}$ is isomorphic to the closed substack of $\underline{\alpha}$ -unstable parabolic bundles.

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2. Parabolic bundles

A *quasi-parabolic bundle* $(E, (f_p)_{p \in I})$ on X with quasi-parabolic structure at I is a vector bundle E on X and flags f_p of the fibre of E over p , for $p \in I$:

$$E_p = E_{p,1} \supset E_{p,2} \supset \cdots \supset E_{p,l_p} \supset E_{p,l_p+1} = 0.$$

The positive integers $n_i(p) = \text{rk}(E_{p,i}/E_{p,i+1})$ are the *multiplicities* of $(E, (f_p)_{p \in I})$ at p and l_p is the *length of the flag* f_p . Let $r_i(p)$ denote $\sum_{j=1}^i n_i(p)$.

This is equivalent to considering filtered locally free sheaves

$$E = E_{(p,1)} \supset E_{(p,2)} \supset \cdots \supset E_{(p,l_p)} \supset E_{(p,l_p+1)} = E(-p),$$

where $E_{(p,i)} = \ker(E \rightarrow E_p \rightarrow E_p/E_{p,i})$. This filtration will again be denoted by f_p . Here the multiplicities are defined as the integers $n_i(p) = \text{deg}(E_{(p,i)}/E_{(p,i+1)})$.

Let $\text{Flag}_{n_1, \dots, n_l}(E_p)$ be the flag variety of E_p of type (n_1, \dots, n_l) . It is an irreducible projective variety of dimension $d_{n_1, \dots, n_l} = \sum_{i>j} n_i n_j$.

Let $E \xrightarrow{q} G$ be a quotient bundle of E . Then a quasi-parabolic structure on E induces a quasi-parabolic structure on G : let $h_{p,i}$ be the injection $E_{(p,i)} \hookrightarrow E$ and denote $G_{(p,i)} = \text{Im}(qh_{p,i})$. Then the quotient morphism induces a filtration at each parabolic point

$$G = G_{(p,1)} \supseteq G_{(p,2)} \supseteq \dots \supseteq G_{(p,l_p)} \supseteq G_{(p,l_p+1)} = G(-p).$$

By considering the distinct locally free sheaves of each filtration, this defines a quasi-parabolic structure on G . It is induced by the one on E in the sense that the morphism q is naturally compatible with the filtrations. Dually, there is a natural induced quasi-parabolic structure on a subbundle $H \xrightarrow{j} E$: if $\pi = \text{coker}(j)$, then it is obtained by letting $H_{(p,i)} = \ker(\pi h_{p,i})$. In other words, $H_{(p,i)} = H \cap E_{(p,i)}$.

Let $(V'', (f_{pV''})_{p \in I})$ (respectively, $(V', (f_p^{V'})_{p \in I})$) denote the quasi-parabolic structure induced by $(E, (f_p)_{p \in I})$ on a quotient bundle V'' (respectively, a subbundle V').

A *parabolic bundle* E_* on X is a quasi-parabolic bundle with, for all $p \in I$, a sequence of real numbers

$$0 \leq \alpha_1(p) < \alpha_2(p) < \dots < \alpha_{l_p}(p) < 1,$$

attached to the flag at p . These numbers are called *parabolic weights*. It is convenient to introduce Simpson's equivalent definition [Si] of a parabolic bundle as a filtered vector bundle. In the notations of [M-Y], [Y], a parabolic bundle is:

- for all $\alpha \in \mathbf{R}$, a locally free sheaf E_α on X and an isomorphism

$$j_\alpha : E_\alpha(-\sum_{p \in I} p) \xrightarrow{\sim} E_{\alpha+1},$$

- for all $\alpha, \beta \in \mathbf{R}$, such that $\alpha \geq \beta$, an injective morphism $i_{E_*}^{\alpha, \beta} : E_\alpha \hookrightarrow E_\beta$, such that the diagram

$$\begin{array}{ccc} E_{\alpha+1} & \xhookrightarrow{i_{E_*}^{\alpha, \alpha+1}} & E_\alpha \\ j_\alpha \uparrow & & \uparrow id \\ E_\alpha(-\sum_{p \in I} p) & \xhookrightarrow{\quad} & E_\alpha \end{array}$$

commutes,

- a sequence of real numbers $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_L < 1$, such that $i_{E_*}^{\alpha, \alpha_i}$ is an isomorphism $E_\alpha \cong E_{\alpha_i}$, for all $\alpha \in]\alpha_{i-1}, \alpha_i]$.

As a convention for a parabolic bundle E_* the sheaf $E = E_0$ is called the underlying vector bundle and for $\alpha \in \mathbf{R}^+$ the morphisms of the parabolic structure will be denoted by $i_E^\alpha := i_{E_*}^{\alpha, 0}$ and $\pi_E^\alpha := \text{coker}(i_E^\alpha)$.

Let E_* and F_* be parabolic bundles on X , with parabolic structure at I . A morphism $\varphi : E \rightarrow F$ is parabolic if, for all $\alpha \in \mathbf{R}^+$, the composition $\pi_F^\alpha \varphi i_E^\alpha$ is the

zero morphism. This produces a morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_\alpha & \xrightarrow{i_E^\alpha} & E & \longrightarrow & E/E_\alpha \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \\ 0 & \longrightarrow & F_\alpha & \longrightarrow & F & \xrightarrow{\pi_F^\alpha} & F/F_\alpha \longrightarrow 0 \end{array}$$

that will be denoted by $\varphi_\alpha : E_\alpha \rightarrow F_\alpha$. The notation $\varphi_* : E_* \rightarrow F_*$, means that φ is parabolic.

Consider the sheaf defined by

$$\mathcal{H}om(E_*, F_*)(\mathcal{U}) = \{\varphi_*^\mathcal{U} : E_*|_\mathcal{U} \rightarrow F_*|_\mathcal{U}\}.$$

By definition of parabolic morphism, it is a subsheaf of $\mathcal{H}om(E, F)$ and for all open subset $\mathcal{U} \subset X$, such that $I \cap \mathcal{U} = \emptyset$, it actually is $\mathcal{H}om(E_*, F_*)(\mathcal{U}) = \mathcal{H}om(E, F)(\mathcal{U})$. Thus the quotient sheaf is a torsion sheaf with support at I that can be described in terms of the parabolic structures of E_* and F_* ; this is done in [B-H], lemma 2.4: suppose for simplicity that $I = \{p\}$ and let $(\alpha_1, \dots, \alpha_l)$ be the weights, $n_i^{E_*} = \deg(E_{\alpha_i}/E_{\alpha_{i+1}})$ the multiplicities of E_* , $(\beta_1, \dots, \beta_h)$ the weights, $n_j^{F_*} = \deg(F_{\beta_j}/F_{\beta_{j+1}})$ the multiplicities of F_* . Then there is a short exact sequence

$$(1) \quad 0 \longrightarrow \mathcal{H}om(E_*, F_*) \longrightarrow \mathcal{H}om(E, F) \longrightarrow \tau_{E_*, F_*} \longrightarrow 0,$$

where τ_{E_*, F_*} is a torsion sheaf supported at p of degree

$$h^0(X, \tau_{E_*, F_*}) = \sum_{\substack{i,j \\ \alpha_i > \beta_j}} n_i^{E_*} n_j^{F_*}.$$

This is a consequence of the fact that a morphism $\varphi : E \rightarrow F$ is parabolic, if and only if the linear map over the parabolic point

$$\varphi_p = (\varphi_{i,j}) : \bigoplus_i E_{\alpha_i}/E_{\alpha_{i+1}} \longrightarrow \bigoplus_j F_{\beta_j}/F_{\beta_{j+1}}$$

is such that $\varphi_{i,j} = 0$, for all $\alpha_i > \beta_j$. Hence the fiber of τ_{E_*, F_*} at p is isomorphic to

$$\bigoplus_{\alpha_i > \beta_j} (E_{\alpha_i}/E_{\alpha_{i+1}})^\vee \otimes F_{\beta_j}/F_{\beta_{j+1}}.$$

For a general parabolic subset I , the degree of the torsion sheaf τ_{E_*, F_*} can be computed as

$$h^0(X, \tau_{E_*, F_*}) = \sum_{p \in I} \sum_{\substack{i,j \\ \alpha_i(p) > \beta_j(p)}} n_i^{E_*}(p) n_j^{F_*}(p).$$

Let $\chi(E_*, F_*)$ denote $\chi(\mathcal{H}om(E_*, F_*))$. By Riemann-Roch formula and the exact sequence (1), this Euler characteristic can be computed as

$$\chi(E_*, F_*) = \text{rk}(E) \deg(F) - \text{rk}(F) \deg(E) + \text{rk}(E) \text{rk}(F)(1 - g) - h^0(\tau_{E_*, F_*}).$$

The group of global sections $H^0(\mathcal{H}om(E_*, F_*))$ is the group of parabolic morphisms, $\text{Hom}(E_*, F_*)$. The first cohomology group $H^1(\mathcal{H}om(E_*, F_*))$ is, by [Y] lemma 1.4, isomorphic to the group of isomorphism classes of parabolic extensions of F_* by E_* and is denoted by $\text{Ext}^1(E_*, F_*)$. By definition, a parabolic extension is a short exact sequence

$$0 \longrightarrow E'_* \xrightarrow{i_*} E_* \xrightarrow{p_*} E''_* \longrightarrow 0,$$

two parabolic extensions being isomorphic, if there is a parabolic isomorphism of extensions.

Recall the definitions of the parabolic invariants of E_* . Let $d = \deg(E)$ and $r = \text{rk}(E)$; the *parabolic degree* of E_* is defined as the real number

$$\deg(E_*) = \deg(E) + \sum_{p \in I} \sum_{i=1}^{l_p} n_i(p) \alpha_i(p)$$

and can be computed as the integral

$$\int_0^1 \deg(E_\alpha) d\alpha + r|I| = \int_{-1}^0 \deg(E_\alpha) d\alpha.$$

The *parabolic Hilbert polynomial* is

$$P(E(m)_*) = \deg(E(m)_*) + r(1 - g) = \deg(E_*) + r(m + 1 - g)$$

and the *parabolic slope* is $\mu(E_*) = \frac{\deg(E_*)}{r}$.

Let E_* be a parabolic bundle and $E \rightarrow G$ a quotient vector bundle. Consider the parabolic structure obtained by the induced quasi-parabolic structure, weighted as the one of E_* . This parabolic structure is said to be the one induced on G by E_* and will be denoted by G_* . Dually, all subbundle $H \hookrightarrow E$ has an induced parabolic structure, that will be denoted by H_* . Recall the notations of the induced quasi-parabolic structure. For all $p \in I$ and $i = 1, \dots, l_p$, consider the integers $n_i''(p) = \deg(G_{(i)}/G_{(i+1)})$. They verify $0 \leq n_i''(p) \leq n_i(p)$ and $n_1''(p) + \dots + n_{l_p}''(p) = \text{rk}(G)$. Then we can easily check the equality

$$\deg(G_*) = \deg(G) + \sum_{p \in I} \sum_{i=1}^{l_p} n_i''(p) \alpha_i(p).$$

Remark 2.1. For all parabolic structure $G_{*'}$ (respectively, $H_{*'}$), such that $E_* \rightarrow G_{*'}$ (respectively, $H_{*'} \hookrightarrow E_*$) is parabolic, it is $\deg(G_*) \leq \deg(G_{*'})$ (respectively, $\deg(H_*) \geq \deg(H_{*'})$).

Definition 2.2. A parabolic bundle E_* is *semistable* if, for all quotient bundle $E \rightarrow G$, the inequality $\mu(E_*) \leq \mu(G_*)$ holds. A *semistable bundle is stable* if the inequality is strict, whenever G is a nontrivial quotient of E .

Suppose, to simplify the formulation of the following basic facts, that $I = \{p\}$.

It is easily seen, that the (semi)stability of a parabolic bundle, as a function of the weights, just depends on their differences. More precisely, let $E_{*'}$ be the bundle with same quasi-parabolic structure as E_* and weights

$$0 \leq 0 < \delta_1 < \dots < \delta_1 + \dots + \delta_{l-1} < 1,$$

where $\delta_i = \alpha_{i+1} - \alpha_i$. Then E_* is (semi)stable if and only if $E_{*'}$ is (semi)stable. Thus we can (and will) assume in the following that the smallest weight at each parabolic point is zero. With the notations of [B-H] this means that we represent the weights in the face $\partial_0 W$ of W .

It will be useful to remark that this assumption allows to write the parabolic degree as

$$\deg(E_*) = \deg(E) + \sum_{p \in I} \sum_{i > j} n_i(p) \delta_j(p).$$

As it is shown in [Me-S], § 2 the (semi)stability condition actually depends on rational weights, *i.e.* there is a rational system of weights $(\alpha'_1, \dots, \alpha'_l)$, such that E_* is (semi)stable if and only if it is (semi)stable with respect to the weights (α'_j) . For this reason, in what follows we consider rational weights.

With the notations of [B-H] for the variation of the (semi)stability condition, for a fixed quasi-parabolic structure there exists an open subset of the space of weights W such that, for any system of weights in this open subset, the condition of semistability is equivalent to the condition of stability. Actually, this open subset is the complement of a union of hyperplanes [Me-S], that we will call *Seshadri walls*. Writing the weights (α_i) as rational numbers $\alpha_i = \frac{a_i}{k}$, these hyperplanes are given by the equations $h = (r, k \deg(E_*))$, for some integer $h \geq 2$.

3. The schemes of quasi-parabolic and parabolic quotients

Let (E, f) be a rank r vector bundle endowed with a quasi-parabolic structure at p of multiplicities (n_1, \dots, n_l) . Consider the set of quotient bundles G of E , whose induced quasi-parabolic structure is of fixed type (n''_1, \dots, n''_l) , that is if (G, f_G) is the induced structure, then $f_G \in \text{Flag}_{n''_1, \dots, n''_l}(G_p)$. In fact this set can be equipped with a natural algebraic structure: in the first part of this section we construct a subscheme of Grothendieck's scheme of quotients, parametrizing quotient bundles of fixed induced multiplicities.

The same construction applies to the case of quotients of a parabolic bundle E_* . This produces a scheme parametrizing quotient bundles of fixed induced parabolic type.

Let $\text{Quot}_{r'', d''}(E)$ be the scheme of quotients of E of rank r'' and degree d'' and let $\text{Quot}_{r'', d''}^o(E)$ denote the open subscheme of quotients $E \xrightarrow{q} G$, such that G is a locally free sheaf. Denote by π_X, π_Q the projections of $X \times \text{Quot}_{r'', d''}(E)$ on X and $\text{Quot}_{r'', d''}^o(E)$ respectively and $\pi_X^*(E) \xrightarrow{\pi} \mathcal{G}$ the universal quotient. The morphism $h_i : E_{(i)} \rightarrow E$ produces an injective morphism $\pi_X^* h_i$ of locally free sheaves on $X \times \text{Quot}_{r'', d''}^o(E)$ and the image of the composition $\pi \pi_X^*(h_i)$ is a subsheaf $\mathcal{G}_{(i)}$ of the universal family of quotients \mathcal{G} :

$$\begin{array}{ccccc} \dots & \longrightarrow & \pi_X^* E_{(3)} & \xrightarrow{\pi_X^*(h_3)} & \pi_X^* E_{(2)} & \xrightarrow{\pi_X^*(h_2)} & \pi_X^* E & \longrightarrow & \dots \\ & & \downarrow \pi_3 & & \downarrow \pi_2 & & \downarrow \pi & & \\ \dots & \longrightarrow & \mathcal{G}_{(3)} & \hookrightarrow & \mathcal{G}_{(2)} & \hookrightarrow & \mathcal{G} & & \dots \end{array}$$

There is a flattening stratification of $\mathcal{G}_{(2)}$ on $X \times \text{Quot}_{r'', d''}^o(E)$. Thus we can write the scheme of locally free quotients as a disjoint union

$$\text{Quot}_{r'', d''}^o(E) = \coprod_{\nu_1=0}^{r''} Q_{\nu_1}$$

where, set theoretically, the scheme Q_{ν_1} consists of those quotients that can be seen as points of $\text{Quot}_{r'', d'' - \nu_1}(E_{(2)})$ via the induced surjective map:

$$Q_{\nu_1} = \{[E \xrightarrow{q} G] \in \text{Quot}_{r'', d''}^o(E) \mid \deg(G_{(2)}) = d'' - \nu_1\}.$$

Restrict the filtration of the universal family to the stratum Q_{ν_1} and consider the sheaf $\mathcal{G}_{(3)}$ as an $\mathcal{O}_{X \times Q_{\nu_1}}$ -module. Then there is a flattening stratification of Q_{ν_1} with respect to the family $\mathcal{G}_{(3)}$, hence we can write:

$$Q_{\nu_1} = \coprod_{\nu_2=0}^{r''} Q_{\nu_1 \nu_2}.$$

Thus, taking flattening stratifications of each stratum, we end up with a stratification of $Q_{\nu_1 \dots \nu_{i-1}}$, with respect to $\mathcal{G}_{(i+1)}$:

$$Q_{\nu_1 \dots \nu_{i-1}} = \coprod_{\nu_i=0}^{r''} Q_{\nu_1 \dots \nu_{i-1} \nu_i}.$$

Remark 3.1. Let ν_1, \dots, ν_i be positive integers such that $\nu_1 + \dots + \nu_i > r''$. Then the stratum $Q_{\nu_1 \dots \nu_i}$ is empty.

This is straightforward, since there is a natural isomorphism $\mathcal{G}_{(l+1)} \xrightarrow{\sim} \mathcal{G}(-p) = \mathcal{G} \otimes \pi_X^* \mathcal{O}_X(-p)$.

Thus, the induced quasi-parabolic type on quotients gives a stratification of Grothendieck's scheme of locally free quotients

$$\text{Quot}_{r'', d''}^o(E) = \coprod_{\nu_1 + \dots + \nu_{l-1} = 0}^{r''} Q_{\nu_1 \dots \nu_{l-1}}.$$

Definition 3.2. Let (n_1'', \dots, n_l'') be integers, such that $0 \leq n_i'' \leq n_i$ and $n_1'' + \dots + n_l'' = r''$. We define the scheme of quasi-parabolic quotients of (E, f) of type (n_1'', \dots, n_l'') as the stratum $Q_{n_1'' \dots n_l''}$ and will denote it by $\text{Quot}_{(n_i''), d''}^{qpar}(E, f)$.

Let $I \subset X$ be a finite subset of points and $(E, (f_p)_{p \in I})$ a quasi-parabolic structure on E . Then quotient bundles of E of rank r'' , degree d'' and fixed induced quasi-parabolic type are parametrized by a locally closed subscheme of Grothendieck's scheme of quotients. Let $((n_i''(p))_{p \in I})$ be integers such that, for all $p \in I$, it is $0 \leq n_i''(p) \leq n_i(p)$, for all $i = 1, \dots, l_p$ and $\sum_i n_i''(p) = r''$. We define the *scheme of quasi-parabolic quotients* of $(E, (f_p)_{p \in I})$ of type $((n_i''(p))_{p \in I})$ as the intersection

$$\text{Quot}_{((n_i''(p))_{p \in I}), d''}^{qpar}(E, (f_p)_{p \in I}) = \bigcap_{p \in I} \text{Quot}_{(n_i''(p)), d''}^{qpar}(E, f_p).$$

Let E_* be a parabolic bundle. The construction of the scheme of quotients of E_* of fixed induced parabolic structure is completely analogue to the construction of the scheme of quasi-parabolic quotients. It is enough to consider flattening stratifications of the families \mathcal{G}_{α_i} :

$$\begin{array}{ccccc} \cdots & \longrightarrow & \pi_X^* E_{\alpha_2} & \xrightarrow{\pi_X^*(i_{E_*}^{\alpha_2, \alpha_1})} & \pi_X^* E_{\alpha_1} & \xrightarrow{\pi_X^*(i_{E_*}^{\alpha_1})} & \pi_X^* E & . \\ & & \downarrow \pi_{\alpha_2} & & \downarrow \pi_{\alpha_1} & & \downarrow \pi & \\ \cdots & \longrightarrow & \mathcal{G}_{\alpha_2} & \hookrightarrow & \mathcal{G}_{\alpha_1} & \hookrightarrow & \mathcal{G} & \end{array}$$

The strata are here determined by the Hilbert polynomials at the weights α_i . This fixes the parabolic Hilbert polynomial of the quotients, since this is determined by the Hilbert polynomials at each weight.

There are more useful multiplicities than the quasi-parabolic ones, that we introduce here, since they better fit to the parabolic filtration. For a point of a stratum, represented by a quotient bundle G_* , let n''_{α_1} be the positive integer defined by $\deg(G_{\alpha_1}) = d'' - n''_{\alpha_1}$ and in general n''_{α_i} the integer such that $\deg(G_{\alpha_i}) = \deg(G_{\alpha_{i-1}}) - n''_{\alpha_i}$. In analogy with the quasi-parabolic multiplicities, we will denote $r''_{\alpha_i} = \sum_{j=1}^i n''_{\alpha_j}$.

By letting $\alpha_0 = 0$ the parabolic degree d''_* can be viewed as a polynomial in the weights, in fact it can be computed as

$$d''_* = d'' + \sum_{i=1}^{L+1} n''_{\alpha_i} \alpha_{i-1}.$$

The parabolic structure induced on a quotient bundle G of E_* is determined by the decreasing step function, that we denote by s_* , associated with the collection of the degrees at each weight, that is the *parabolic degree function*

$$\alpha \mapsto s_\alpha = \deg(G_\alpha).$$

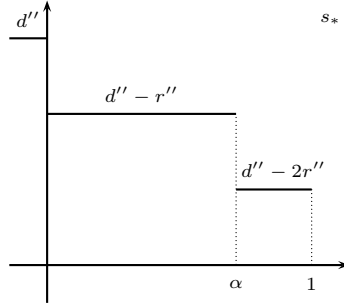
Remark that, if s_* and s'_* are the parabolic degree functions of quotient bundles V and W of E_* of same rank r'' , for all $\beta \in \mathbf{R}$ it is $s_{1+\beta} = s_\beta - r''|I|$ and the same holds for s'_* , hence the function $s_* - s'_*$ is periodic of period 1 and

$$\int_{\beta}^{\beta+1} s_\alpha - s'_\alpha d\alpha = \deg(V_*) - \deg(W_*).$$

Definition 3.3. We define the scheme of parabolic quotients of E_* of rank r'' and induced parabolic type s_* as the stratum corresponding to the parabolic degree function s_* and will denote it by $\text{Quot}_{r'', s_*}^{\text{par}}(E_*)$.

Remark 3.4. This construction allows to consider an algebraic structure on a scheme parametrizing parabolic quotient bundles, with possibly different underlying quasi-parabolic structures.

For instance, consider a parabolic bundle E_* on X at $I = \{p, q\}$ of rank r and weight $0 < \alpha_1(p) = \alpha_1(q) = \alpha < 1$ and suppose its underlying quasi-parabolic structure is such that $n_i(p), n_i(q) \geq r''$, for $i = 1, 2$. Let s_* be a parabolic degree function of quotient bundles of E of rank r'' and degree d'' , defined by $s_0 = d''$ and $s_\alpha = d'' - r''$.



Then the function s_* corresponds to $r'' + 1$ quasi-parabolic structures, *i.e.* as a set $\text{Quot}_{r'', s_*}^{\text{par}}(E_*)$ is the following disjoint union

$$\prod_{k=0}^{r''} \text{Quot}_{(k, r''-k), d''}^{\text{qpar}}(E, f_p) \cap \text{Quot}_{(r''-k, k), d''}^{\text{qpar}}(E, f_q).$$

Let $d_{r'', *}$ denote the minimal parabolic degree of a rank r'' quotient bundle of E . We denote by \bar{s}_* a parabolic degree function whose parabolic degree is $d_{r'', *}$, that is $\int_{-1}^0 \bar{s}_\alpha d\alpha = d_{r'', *}$.

For a parabolic structure $*$ on E , we denote by $*'$ a parabolic structure obtained from $*$ by dropping one weight. For instance, consider the parabolic structure obtained by dropping the highest weight α_L :

$$E \supset E_{\alpha_1} \supset E_{\alpha_2} \supset \cdots \supset E_{\alpha_{L-1}} \supset E(-\sum_{p \in I} p).$$

Then a parabolic degree function s_* for quotient bundles of rank r'' completely determines the parabolic degree function for the structure $*'$ and we denote it by s'_{*} . This means that the stratum $\text{Quot}_{r'', s_*}^{\text{par}}(E_*)$ naturally is a substratum of the scheme $\text{Quot}_{r'', s'_{*}}^{\text{par}}(E_{*'})$. Remark that the parabolic degree of these strata are such that

$$(2) \quad \int_{-1}^0 s'_\alpha d\alpha = \int_{-1}^0 s_\alpha d\alpha - (\alpha_L - \alpha_{L-1})n''_{\alpha_{L+1}}.$$

Remark 3.5. Consider a parabolic degree function s_* for which the last multiplicity is maximal, that is $n''_{\alpha_{L+1}} = \min\{n_{\alpha_{L+1}}, r''|I|\}$. Let d''_* denote the parabolic degree of this stratum and $d''_{*'}$ denote the parabolic degree of the stratum s'_{*} . Then equality (2) is

$$d''_{*'} = d''_* - (\alpha_L - \alpha_{L-1})n''_{\alpha_{L+1}}.$$

Let $E \rightarrow V$ be a quotient vector bundle of E of rank r'' . Denote by $\bar{d}_* = \deg(V_*)$ and $\bar{d}_{*'} = \deg(V_{*'})$ the parabolic degrees with respect to the structures induced by $*$ and $*'$ respectively. Then there exists an integer n for which equality (2) can be written as

$$\bar{d}_{*'} = \bar{d}_* - (\alpha_L - \alpha_{L-1})n.$$

Note that, since we assume that $n''_{\alpha_{L+1}}$ is maximal, it is $n''_{\alpha_{L+1}} - n \leq 0$. This translates into the following inequality, on the differences of parabolic degrees

$$(3) \quad d''_{*'} - \bar{d}_{*'} = d''_* - \bar{d}_* - (\alpha_L - \alpha_{L-1})(n''_{\alpha_{L+1}} - n) \leq d''_* - \bar{d}_*.$$

In particular, if V is in a stratum corresponding to the minimal parabolic degree $d_{r'',*}$, inequality (3) and the fact that for all quotient bundle V it is $\deg(V_*) \leq d_{r'',*}$, imply the following inequality

$$d''_{*'} - d_{r'',*'} = d''_{*'} - \deg(V_{*'}) \leq d''_* - \deg(V_*) \leq d''_* - d_{r'',*}.$$

We still denote by d''_* the parabolic degree of the stratum s_* . We want to prove the following estimate for the dimension of the parabolic strata.

Theorem 3.6. *With the notations above, it is*

$$\dim(\text{Quot}_{v_{r'',s_*}}^{\text{par}}(E_*)) \leq r''(r - r'') + r(d''_* - d_{r'',*}).$$

Remark 3.7. This estimate depends on the parabolic invariants of the stratum. In Appendix B we study the example of rank 2 parabolic bundles and show how, under some hypotheses, it is possible to get an estimate depending on the invariants of the underlying vector bundles of the stratum.

Proof: The proof goes by induction on the number of weights. For $L = 0$ the statement is given by the estimate of Popa and Roth [Po-Ro], theorem 4.1 on the dimension of Grothendieck's scheme of quotients.

We have to prove that the statement holds for L weights, provided it holds for $L - 1$ weights. We prove it in two steps. The first one consists in proving the statement for L weights and under the assumption that $n''_{\alpha_{L+1}}$ is maximal. The second step consists in drawing the general case from the first step.

First step

Consider the parabolic structure $*'$ obtained from $*$ by dropping α_L . We still denote by s'_* the parabolic degree function induced by s_* with respect to the structure $*'$. Then there is an obvious inequality

$$\dim(\text{Quot}_{v_{r'',s_*}}^{\text{par}}(E_*)) \leq \dim(\text{Quot}_{v_{r'',s'_*}}^{\text{par}}(E_{*'}))$$

and since the parabolic structure $*'$ has $L - 1$ weights, the statement for the estimate of the dimension of $\text{Quot}_{v_{r'',s'_*}}^{\text{par}}(E_{*'})$ holds by the induction hypothesis. From this inequality and remark 3.5 it follows

$$\begin{aligned} & \dim(\text{Quot}_{v_{r'',s_*}}^{\text{par}}(E_*)) \\ & \leq \dim(\text{Quot}_{v_{r'',s'_*}}^{\text{par}}(E_{*'})) \leq r''(r - r'') + r(d''_{*'} - d_{r'',*'}) \\ & \leq r''(r - r'') + r(d''_* - d_{r'',*}). \end{aligned}$$

This proves the statement for the strata of a parabolic structure with L weights, whose last multiplicity is maximal.

Remark 3.8. We have actually proved that the statement holds for all strata for which there is an $i \in \{1, \dots, L + 1\}$ such that n''_{α_i} is maximal. This is due to the fact that the parabolic strata are obtained by successive flattening stratifications

We have the inclusion

$$\mathrm{Quot}_{r'', \tilde{s}_*}^{\mathrm{par}}(E_*) \subseteq \mathrm{Quot}_{r'', \tilde{s}'_*}^{\mathrm{par}}(\tilde{E}_{\tilde{s}'_*}).$$

From this inclusion and lemma 3.9 we get the inequality

$$\dim(\mathrm{Quot}_{r'', s_*}^{\mathrm{par}}(E_*)) = \dim(\mathrm{Quot}_{r'', \tilde{s}_*}^{\mathrm{par}}(E_*)) \leq \dim(\mathrm{Quot}_{r'', \tilde{s}'_*}^{\mathrm{par}}(\tilde{E}_{\tilde{s}'_*})).$$

By the first step of the induction argument, the last dimension is less than or equal to

$$r''(r - r'') + r(d''_{\tilde{s}'_*} - d_{r'', \tilde{s}'_*}).$$

Let \tilde{s}'_* be a parabolic degree function realizing the minimal parabolic degree d_{r'', \tilde{s}'_*} . By remark 3.5, it is

$$d''_{\tilde{s}'_*} - d_{r'', \tilde{s}'_*} \leq d''_* - \bar{d}_*,$$

where \bar{d}_* is the parabolic degree of some vector bundle \bar{V} of the stratum \tilde{s}'_* . Moreover, if we denote by \bar{d}_* the parabolic degree of \bar{V} with respect to the original parabolic structure $*$, since the structure $\tilde{*}$ has one weight more than $*$, it is $\bar{d}_* \geq \bar{d}_*$. Thus we get

$$d''_{\tilde{s}'_*} - d_{r'', \tilde{s}'_*} \leq d''_* - \bar{d}_* \leq d''_* - \bar{d}_*.$$

Recalling the expression of d''_* , we get the inequality

$$\begin{aligned} \dim(\mathrm{Quot}_{r'', s_*}^{\mathrm{par}}(E_*)) &\leq r''(r - r'') + r(d''_* + \tilde{\alpha}n''_{\alpha_1} - \bar{d}_*) \\ &\leq r''(r - r'') + r(d''_* - d_{r'', *}) + \tilde{\alpha}n''_{\alpha_1}. \end{aligned}$$

Then in order to get the inequality of the statement it is enough to remark that, according to lemma 3.9, we can choose $\tilde{\alpha}$ as small as we like. \square

Proof of lemma 3.9: It will be enough to prove that

$$\dim(\mathrm{Quot}_{r'', s_*}^{\mathrm{par}}(E_*)) \leq \dim(\mathrm{Quot}_{r'', \tilde{s}_*}^{\mathrm{par}}(E_{\tilde{s}_*})),$$

that is, to find some subscheme of the substratum associated with \tilde{s}_* with same dimension as the stratum associated with s_* . Let Q be an irreducible component of $\mathrm{Quot}_{r'', s_*}^{\mathrm{par}}(E_*)$ of maximal dimension and let V be a quotient bundle of E , representing a point of Q . By assumption on the function s_* it is

$$\deg(E_{\alpha_1}/E(-\sum_{p \in I} p)) = n_{\alpha_2} + \cdots + n_{\alpha_{L+1}} >$$

$$\deg(V_{\alpha_1}/V(-\sum_{p \in I} p)) = n''_{\alpha_2} + \cdots + n''_{\alpha_{L+1}}$$

and moreover, since $n''_{\alpha_1} > 0$, it is $n''_{\alpha_2} + \cdots + n''_{\alpha_{L+1}} < r''|I|$. Translating this into quasi-parabolic conditions, this means that there are some $p \in I$ such that $\alpha_1(p) = \alpha_1$ and at p the induced parabolic structure is a strict inclusion in the fiber of V :

$$\begin{array}{ccc} E_p & \xrightarrow{q} & V_p \\ \uparrow i & & \uparrow \\ E_{p,2} & \xrightarrow{\quad} & V_{p,2} = \mathrm{Im}(qi) \end{array}$$

On the other hand, we can add the missing generators of V_p at such points: we can choose a subset $I' \subseteq I$ of points such that $\alpha_1(p) = \alpha_1$ and at these points add a linear subspace $H_{p,2}$ to $E_{p,2}$, with $\mathrm{rk}(H_{p,2}) = \mathrm{rk}(V_p/V_{p,2})$ for which the composition

$$H_{p,2} \oplus E_{p,2} \hookrightarrow E_p \twoheadrightarrow V_p$$

has rank r'' and $\sum_{p \in I'} \text{rk}(H_{p,2}) = n''_{\alpha_1}$. It is enough to choose the vector space $H_{p,2}$ as the image of a section of the surjective linear map

$$E_p \twoheadrightarrow V_p/V_{p,2}.$$

The parabolic structure $\tilde{*}$ is obtained by enriching the flags of the quasi-parabolic structures at the points of I' in the following way:

$$E_p \supset H_{p,2} \oplus E_{p,2} \supset E_{p,2} \supset \cdots \supset E_{p,l_p} \supset 0$$

with weights $(\alpha_i(p)) = (\tilde{\alpha}, \alpha_1(p), \dots, \alpha_{l_p}(p))$. This means that V represents a point of the substratum $\text{Quot}_{r'', \tilde{s}_*}^{\text{par}}(E_{\tilde{*}})$. All is left to check is that this is true in an open neighbourhood of V in Q .

By semi-continuity of the rank, the composed linear map

$$H_{p,2} \oplus E_{p,2} \hookrightarrow E_p \twoheadrightarrow W$$

has rank r'' , for all vector spaces W in an open neighbourhood U_p of the isomorphism class of the fiber V_p in the grassmannian $G_p = \text{Grass}_{r''}(E_p)$. Hence the induced quasi-parabolic filtration is of the same type, for all vector bundle in an open neighbourhood of V . Recall that all the points of the parabolic strata are vector bundles and consider, for all $p \in I'$, the map

$$\epsilon_p : Q \rightarrow G_p, \quad [E \xrightarrow{\pi} W] \mapsto [E_p \xrightarrow{\pi_p} W_p].$$

Let Q' be the open subscheme of Q defined as the intersection

$$Q' = \bigcap_{p \in I'} \epsilon_p^{-1}(U_p).$$

Since Q is irreducible, we have $\dim(Q') = \dim(Q)$ and by construction Q' is a subscheme of $\text{Quot}_{r'', \tilde{s}_*}^{\text{par}}(E_{\tilde{*}})$. \square

4. Sections of the line bundle \mathcal{L}^{par}

Let \mathcal{M}^{par} denote the moduli space of parabolic bundles on X of rank r , trivial determinant and parabolic structure at I of multiplicities $((n_1(p), \dots, n_{l_p}(p))_{p \in I})$ and weights

$$0 \leq \alpha_1(p) = 0 < \alpha_2(p) < \alpha_3(p) < \cdots < \alpha_{l_p}(p) < 1.$$

Let $((d_1(p), \dots, d_{l_p-1}(p))_{p \in I}, k)$ be strictly positive integers such that, for all $j =$

$2, \dots, l_p$, the weights at p can be written as $\alpha_j(p) = \frac{1}{k} \sum_{h=1}^{j-1} d_h(p)$.

A family \mathcal{E}_* of parabolic bundles at I , of rank r , trivial determinant, multiplicities $((n_i(p))_{p \in I})$ and weights $((d_j(p))_{p \in I}, k)$ parametrized by a scheme S is a vector bundle \mathcal{E} over $X \times S$ of rank r , such that $\det(\mathcal{E}) = \mathcal{O}_{X \times S}$ and, for all $p \in I$, quotient bundles $Q_i(p)$ of $\mathcal{E}_{\{p\} \times S}$, of rank $r_i(p) = n_1(p) + \cdots + n_i(p)$, such that, by letting $\mathcal{K}_i(p)$ denote the kernel

$$0 \longrightarrow \mathcal{K}_i(p) = \ker(\pi_i(p)) \longrightarrow \mathcal{E}_{\{p\} \times S} \xrightarrow{\pi_i(p)} Q_i(p) \longrightarrow 0,$$

then $\mathcal{K}_i(p) \subset \mathcal{K}_{i-1}(p)$ for all $i = 1 \dots, l_p - 1$. The family is parabolic in the sense that, for all $s \in S$ the vector bundle \mathcal{E}_s , has the quasi-parabolic structure

$$\mathcal{E}_{s,p} \supset \mathcal{K}_1(p)_s \supset \cdots \supset \mathcal{K}_{l-1}(p)_s \supset 0$$

and weights $((d_j(p))_{p \in I}, k)$. Actually the family \mathcal{E} has a weighted filtration, induced by the quotients $Q_i(p)$, that is

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_{\delta_1(p)} \supset \cdots \supset \mathcal{E}_{\delta_1(p) + \cdots + \delta_{i_{p-1}}(p)} \supset \mathcal{E}_1 = \mathcal{E} \otimes \pi_X^* \mathcal{O}_X(-p),$$

where $\delta_j(p) = \frac{d_j(p)}{k}$ and $Q_j(p) \cong \mathcal{E} / \mathcal{E}_{\delta_1(p) + \cdots + \delta_j(p)}$. Suppose that the family is semistable and let $\phi_S : S \rightarrow \mathcal{M}^{par}$ the modular morphism. Suppose that $\sum_{p \in I} \sum_{i > j} \frac{n_i(p) d_j(p)}{r} \in \mathbf{Z}$ and let $\mathcal{L}^{par}(\mathcal{E}_*)$ be the line bundle on S defined as the tensor product

$$\mathcal{L}^{par}(\mathcal{E}_*) = (\det R\pi_S \mathcal{E})^{\otimes k} \otimes \bigotimes_{p \in I} \bigotimes_j (\det Q_j(p))^{\otimes d_j(p)} \otimes (\det \mathcal{E}_{|\{q\} \times S})^{\otimes e}.$$

Here e is an integer depending on the parabolic structure defined by

$$e = \frac{1}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p) + k(1 - g) - \sum_{p \in I} \sum_j d_j(p).$$

Theorem 4.1. ([N-R], theorem 1, for rank 2; [Pa1], theorem 3.3, for arbitrary rank) *There exists a unique ample line bundle \mathcal{L}^{par} over \mathcal{M}^{par} such that, for all semistable parabolic family \mathcal{E}_* parametrized by S , it is $\phi_S^* \mathcal{L}^{par} = \mathcal{L}^{par}(\mathcal{E}_*)$.*

In the case of rank two parabolic bundles, Pauly gives in [Pa2] a method to produce sections of \mathcal{L}^{par} of type theta. In what follows, we extend his method to the rank r case and produce sections of $\mathcal{L}^{par \otimes h}$, for all $h \in \mathbf{N}$.

Let $\mathcal{E}_*, \mathcal{F}_*$ be families of parabolic bundles on I , parametrized by S , of quotients and flags respectively

$$\begin{aligned} 0 \longrightarrow \mathcal{E}_\alpha \xrightarrow{i_\alpha} \mathcal{E} \xrightarrow{p_\alpha} Q_\alpha \longrightarrow 0 & \quad 0 \rightarrow \mathcal{K}_i(p) \rightarrow \mathcal{E}_{|\{p\} \times S} \rightarrow Q_i(p) \rightarrow 0, \\ 0 \longrightarrow \mathcal{F}_\alpha \xrightarrow{i'_\alpha} \mathcal{F} \xrightarrow{p'_\alpha} Q'_\alpha \longrightarrow 0 & \quad 0 \rightarrow \mathcal{K}'_j(p) \rightarrow \mathcal{F}_{|\{p\} \times S} \rightarrow Q'_j(p) \rightarrow 0. \end{aligned}$$

A morphism $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ of vector bundles is parabolic if the composed map $p'_\alpha \varphi i_\alpha$ is the zero morphism, for all $\alpha \in \mathbf{R}^+$. The sheaf of parabolic homomorphisms is a locally free subsheaf of $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, that will be denoted by $\mathcal{H}om(\mathcal{E}_*, \mathcal{F}_*)$. The quotient sheaf coker $(\mathcal{H}om(\mathcal{E}_*, \mathcal{F}_*) \hookrightarrow \mathcal{H}om(\mathcal{E}, \mathcal{F}))$ is a family of torsion sheaves parametrized by S whose support is contained in the parabolic subset I , that we denote by $T_{\mathcal{E}_*, \mathcal{F}_*}$.

The sheaf $\mathcal{H}om(\mathcal{E}_*, \mathcal{F}_*)$ is the family on S parametrizing the sheaves of parabolic morphisms between bundles of the families, that is for all $s \in S$ there is a natural isomorphism

$$\mathcal{H}om(\mathcal{E}_{*s}, \mathcal{F}_{*s}) \cong \mathcal{H}om(\mathcal{E}_*, \mathcal{F}_*)_s.$$

Let F be a vector bundle on X of rank hk such that

$$\deg(F) = \frac{h}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p) + hk(g - 1)$$

and let F_* be a parabolic structure at I of multiplicities

$$((hd_1(p), \dots, hd_{i_{p-1}}(p), hk - h \sum_j d_j(p))_{p \in I})$$

and weights $((d_j(p))_{p \in I}, k)$. Let $\pi_X^* F_*$ be the constant family of parabolic bundles of value F_* , parametrized by S . For $s \in S$, let \mathcal{E}_{*s} denote the parabolic bundle of \mathcal{E}_* over s . Then it is

$$\begin{aligned} \chi(\mathcal{E}_{*s}, F_*) &= r h k (\mu(\mathcal{E}_{*s}) + (g-1)) - h k \deg(\mathcal{E}_s) + r h k (1-g) - \sum_{p \in I} \sum_{i > j} n_i^{\mathcal{E}_{*s}}(p) n_j^{F_*}(p) \\ &= h \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p) - \sum_{p \in I} \sum_{i > j} n_i(p) h d_j(p) = 0. \end{aligned}$$

Fix a basis of $F_p = \bigoplus_j F_{p,j}/F_{p,j+1}$ and let $\mathcal{T}_{\mathcal{E}_*, F_*}$ denote the family of torsion sheaves of the short exact sequence of parabolic morphisms

$$(4) \quad 0 \rightarrow \mathcal{H}om(\mathcal{E}_*, \pi_X^* F_*) \rightarrow \mathcal{H}om(\mathcal{E}, \pi_X^* F) \xrightarrow{P} \mathcal{T}_{\mathcal{E}_*, F_*} \rightarrow 0.$$

Lemma 4.2. *With the notations above, it is*

$$\det R\pi_S \mathcal{H}om(\mathcal{E}_*, \pi_X^* F_*) \cong \mathcal{L}^{par}(\mathcal{E}_*)^{\otimes h}.$$

Proof: By the short exact sequence (4) there is a natural isomorphism

$$\det R\pi_S \mathcal{H}om(\mathcal{E}_*, \pi_X^* F_*) \cong (\det R\pi_S \mathcal{H}om(\mathcal{E}, \pi_X^* F)) \otimes (\det R\pi_S \mathcal{T}_{\mathcal{E}_*, F_*})^\vee.$$

By Serre duality theorem, [Pa2], lemma 3.4, there is an isomorphism

$$\det R\pi_S \mathcal{H}om(\mathcal{E}, \pi_X^* F) = \det R\pi_S \mathcal{E}^\vee \otimes \pi_X^* F \cong \det R\pi_S \mathcal{E} \otimes \pi_X^*(F^\vee \otimes K_X).$$

The vector bundle $\det \mathcal{E}_{|\{q\} \times S}$ is independent of $q \in X$ and by [Pa2], lemma 3.5 it follows that

$$\det R\pi_S \mathcal{E} \otimes \pi_X^*(F^\vee \otimes K_X) \cong (\det R\pi_S \mathcal{E})^{\otimes h k} \otimes (\det \mathcal{E}_{|\{q\} \times S})^{\otimes -\deg(F^\vee \otimes K_X)}.$$

Now, since the degree of $F^\vee \otimes K_X$ can be computed as

$$\deg(F^\vee \otimes K_X) = h k \deg(K_X) - \deg(F) = h k (g-1) - \frac{h}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p),$$

the first determinant bundle is isomorphic to

$$\begin{aligned} \det R\pi_S \mathcal{H}om(\mathcal{E}, \pi_X^* F) &\cong (\det R\pi_S \mathcal{E})^{\otimes h k} \\ &\otimes (\det \mathcal{E}_{|\{q\} \times S})^{\otimes \frac{h}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p) + h k (1-g)}. \end{aligned}$$

The sheaf $\mathcal{T}_{\mathcal{E}_*, F_*}$ is a family of skyscraper sheaves supported at I , hence the sheaf $R^1 \pi_{S*} \mathcal{T}_{\mathcal{E}_*, F_*}$ is zero and there is an isomorphism

$$\pi_{S*} \mathcal{T}_{\mathcal{E}_*, F_*} \cong \bigoplus_{p \in I} \bigoplus_{j=1}^{l_p-1} \mathcal{K}_j(p)^\vee \otimes \mathcal{O}_S^{h d_j(p)}.$$

Thus the second determinant can be computed as follows

$$\begin{aligned} (\det R\pi_S \mathcal{T}_{\mathcal{E}_*, F_*})^\vee &\cong \det \bigoplus_{p \in I} \bigoplus_{j=1}^{l_p-1} \mathcal{K}_j(p)^\vee \otimes \mathcal{O}_S^{hd_j(p)} \cong \bigotimes_{p \in I} \bigotimes_{j=1}^{l_p-1} \det(\mathcal{K}_j(p)^\vee \otimes \mathcal{O}_S^{hd_j(p)}) \\ &\cong \bigotimes_{p \in I} \bigotimes_{j=1}^{l_p-1} (\det \mathcal{K}_j(p)^\vee)^{\otimes hd_j(p)}. \end{aligned}$$

By definition, it is $\mathcal{K}_i(p) = \ker(\pi_i(p))$ and this yields the isomorphism

$$\bigotimes_{p \in I} \bigotimes_{j=1}^{l_p-1} (\det \mathcal{K}_j(p)^\vee)^{\otimes hd_j(p)} \cong \bigotimes_{p \in I} \bigotimes_{j=1}^{l_p-1} (\det Q_j(p))^{\otimes hd_j(p)} \otimes (\det \mathcal{E}_{\{p\} \times S}^\vee)^{\otimes hd_j(p)}.$$

The lemma then follows from the fact that for all $p \in I$ there is a natural isomorphism $\det \mathcal{E}_{\{q\} \times S} \cong \det \mathcal{E}_{\{p\} \times S}$ and the equality

$$he = \frac{h}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p) + hk(1-g) - h \sum_{p \in I} \sum_j d_j(p).$$

□

Let \mathcal{F} be a quasi-coherent $\mathcal{O}_{X \times S}$ -module, flat over S . Recall how one obtains a complex that is quasi-isomorphic to $R\pi_S \mathcal{F}$. By the relative version of Serre A theorem, there is an integer m_0 such that, if $m \geq m_0$, the natural evaluation morphism

$$K_0 = \pi_X^* \mathcal{O}_X(-m) \otimes \pi_X^* \pi_{X*} \mathcal{F}(m) \xrightarrow{q} \mathcal{F}$$

is surjective. Since $\deg(\mathcal{O}_X(-m)) < 0$, it is $\pi_{S*} K_0 = 0$ and if we denote by $K_1 = \ker q$ it is $\pi_{S*} K_1 = 0$ as well. Moreover the higher direct image sheaves $\mathcal{L}_1 = R^1 \pi_{S*} K_1$, $\mathcal{L}_0 = R^1 \pi_{S*} K_0$ are locally free. Hence the short exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i} K_0 \xrightarrow{q} \mathcal{F} \longrightarrow 0$$

yields the long exact sequence in cohomology

$$0 \longrightarrow \pi_{S*} \mathcal{F} \longrightarrow R^1 \pi_{S*} K_1 \xrightarrow{R^1 \pi_{S*} i} R^1 \pi_{S*} K_0 \longrightarrow R^1 \pi_{S*} \mathcal{F} \longrightarrow 0$$

and there is a natural isomorphism $\det R\pi_S \mathcal{F} \cong \det \mathcal{L}_0 \otimes \det \mathcal{L}_1^\vee$.

Let $0 \rightarrow \mathcal{L}_1 \xrightarrow{\nu} \mathcal{L}_0 \rightarrow 0$ be a complex of locally free sheaves on $X \times S$, quasi-isomorphic to $R\pi_S \mathcal{H}om(\mathcal{E}_*, \pi_X^* F_*)$. The hypothesis on the Euler characteristic $\chi(\mathcal{E}_{*s}, F_{*s}) = 0$, for all $s \in S$, is equivalent to the assumption that the locally free sheaves \mathcal{L}_i have same rank and the morphism of vector bundles $\det \nu : \det \mathcal{L}_1 \rightarrow \det \mathcal{L}_0$ defines a section of $(\det \mathcal{L}_1)^\vee \otimes \det \mathcal{L}_0 = \det R\pi_S \mathcal{H}om(\mathcal{E}_*, \pi_X^* F_*) \cong \mathcal{L}^{par}(\mathcal{E}_*)^{\otimes h}$, that we denote by $\theta_{F_*}^{\mathcal{E}_*}$. This section is zero at a point $s \in S$, if and only if

$$\dim(\mathrm{Hom}(\mathcal{E}_{*s}, F_{*s})) = \dim(\mathrm{Ext}^1(\mathcal{E}_{*s}, F_{*s})) \neq 0.$$

To show that this produces a section of the line bundle on the moduli space \mathcal{M}^{par} , recall its construction (see, for instance, [Pa1], theorem 2.3). Let Q be the scheme of quotients of rank r and trivial determinant of $\mathcal{O}_X^{P(n)}(-n)$, where P is the Hilbert polynomial of such quotients and n is an integer, $n \gg 0$. Let Ω denote the

open subset of Q of locally free quotients, \mathcal{F} the universal family of quotients on $X \times \Omega$ and F_p the flag varieties bundle of multiplicities $(n_1(p), \dots, n_{l_p}(p))$

$$F_p = \mathcal{F} \text{lag}_{(n_1(p), \dots, n_{l_p}(p))}(\mathcal{F}|_{\{p\} \times \Omega}) \xrightarrow{\pi(p)} \Omega.$$

Let $\mathcal{Q}_i(p)$ denote the universal quotients on F_p and let \mathcal{R} be the fibred product of the F_p 's, for $p \in I$, over Ω . We still denote by \mathcal{F}_* and $\mathcal{Q}_i(p)$ the universal families obtained by pullback to \mathcal{R} . The parabolic family \mathcal{F}_* , with parabolic quotients $(\mathcal{Q}_i(p))$, is locally a universal family of parabolic bundles. Let \mathcal{R}^{ss} be the open subscheme of \mathcal{R} of semistable parabolic bundles. Then \mathcal{M}^{par} is obtained as the good quotient of \mathcal{R}^{ss} for the natural action of $SL(P(n))$.

Consider the line bundle $\mathcal{L}^{par}(\mathcal{F}_*)$ on \mathcal{R}^{ss} . By [Pa1], theorem 3.3 it descends to the moduli space \mathcal{M}^{par} . The section $\theta_{F_*}^{\mathcal{F}_*}$ is $SL(P(n))$ -invariant, thus it descends to a section of $\mathcal{L}^{par \otimes h}$, that will be called *parabolic theta function* (of order h) associated with the parabolic bundle F_* .

5. Zeroes of parabolic theta functions

Let E_* be a semistable parabolic bundle on X at I , of rank r , trivial determinant, multiplicities $((n_i(p))_{p \in I})$ and weights $((d_j(p))_{p \in I}, k)$. For a parabolic bundle F_* on X at I , of rank ℓk , slope $\mu(E_*) + g - 1$, multiplicities $((\ell d_1(p), \dots, \ell d_{l_p-1}(p), \ell(k - \sum_{i=1}^{l_p-1} d_i(p)))_{p \in I})$ and same weights as E_* , the parabolic theta function associated with F_* is zero at the point of \mathcal{M}^{par} represented by E_* , if and only if $\text{Hom}(E_*, F_*) = \text{H}^0(\text{Hom}(E_*, F_*)) \neq \{0\}$. Let $d_{l_p}(p) = k - \sum_{i=1}^{l_p-1} d_i(p)$ and let $\mathcal{M}'_{\ell}{}^{par}$ denote the moduli space of equivalence classes of semistable parabolic bundles F_* , with which we can associate parabolic theta functions of order ℓ . Recall that its dimension is given by

$$\dim(\mathcal{M}'_{\ell}{}^{par}) = (\ell k)^2(g-1) + \sum_{p \in I} d_{\ell d_1(p), \dots, \ell d_{l_p}(p)} + 1.$$

Let r'' be an integer such that $0 < r'' \leq r$ and let $\mathcal{E}_{r''}$ denote the family of isomorphism classes of stable parabolic bundles F_* such that there is a morphism $\varphi_* : E_* \rightarrow F_*$ of rank r'' . We prove in this section that whenever $\ell \geq r''(r - r'')$ and $\ell \geq \frac{r}{k}$, then

$$\dim(\mathcal{E}_{r''}) \leq (\ell k)^2(g-1) + \sum_{p \in I} d_{\ell d_1(p), \dots, \ell d_{l_p}(p)}.$$

This will prove theorem 1.1 since if $I \neq \emptyset$ then $k \geq 2$ and

$$\sup_{0 < r'' \leq r} \left\{ r''(r - r''), \frac{r}{k} \right\} \leq \left[\frac{r^2}{4} \right].$$

Then there exists a nonempty open subset \mathcal{U} of the moduli space $\mathcal{M}'_{\ell}{}^{par}$, such that for all F_* representing a point of \mathcal{U} it is $\text{Hom}(E_*, F_*) = 0$.

5.1. Images of parabolic morphisms.

Let $\varphi_* : E_* \rightarrow F_*$ be a morphism. The image of φ is a quotient bundle of E , denote it by $V = \text{Im}(\varphi)$ and let V_* be the induced parabolic structure via the quotient morphism $E_* \rightarrow V$. The subbundle V' of F generated by V inherits a natural parabolic structure as well, via the injective morphism to F_* . We want to compare these two induced parabolic structures. Note that, if the support of the quotient sheaf V'/V does not intersect the parabolic subset I , the two parabolic structures necessarily have the same multiplicities.

Suppose for simplicity that $I = \{p\}$ and let $n'_i = \deg(V'_{\alpha_{i-1}}/V'_{\alpha_i})$ be the multiplicities of the parabolic structure induced on V' by F_* and $n''_i = \deg(V_{\alpha_{i-1}}/V_{\alpha_i})$ be the multiplicities induced on V by E_* .

Proposition 5.1. *With these notations, it is*

$$(5) \quad \deg(V) + \frac{1}{k} \sum_j (r'' - r'_j) d_j \leq \deg(V') + \frac{1}{k} \sum_j (r'' - r'_j) d_j.$$

In particular, the following inequality holds

$$\deg(V_*) \leq \deg(V'_*).$$

Proof: We are actually going to prove that $\deg(V/V_{\alpha_i}) \geq \deg(V'/V'_{\alpha_i})$, for all $i = 1, \dots, l$. In fact, this inequality can be rewritten as

$$r''_i = \sum_{j \leq i} n''_j \geq r'_i = \sum_{j \leq i} n'_j,$$

for all i . The underlying vector bundle V' is the saturation of V in F and so $\deg(V) \leq \deg(V')$. From these facts inequalities (5) follow.

Let $i \in \{1, \dots, l\}$ and denote $\alpha_i = \alpha$. The morphism φ is parabolic, so $V_\alpha \cong \text{Im}(\varphi_\alpha)$ and the diagram (6) commutes. From this we deduce the commutative diagram (7), hence a morphism $j_\alpha : V_\alpha \rightarrow V' \times_F F_\alpha \cong V'_\alpha$.

(6)

(7)

$$(9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & V & \longrightarrow & V' & \longrightarrow & \tau & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow v & & \\ 0 & \longrightarrow & V_\alpha & \longrightarrow & V'_\alpha & \longrightarrow & \tau_\alpha & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow u & & \\ 0 & \longrightarrow & V(-p) & \longrightarrow & V'(-p) & \longrightarrow & \tau & \longrightarrow & 0 \end{array}$$

□

Remark 5.2. Use the notations for the multiplicities introduced at the end of the second section, to construct the scheme of parabolic quotients. Then the same proof shows the following inequalities:

$$(10) \quad \deg(V_*) \leq \deg(V) + \deg(V'_*) - \deg(V') \leq \deg(V'_*).$$

5.2. Parabolic extensions.

Let \mathcal{F}'_* , \mathcal{F}''_* be families of parabolic bundles, parametrized by a scheme S . We want to describe a parameter space for isomorphism classes of nonsplitting parabolic extensions of type

$$0 \longrightarrow \mathcal{F}'_{*s} \longrightarrow \mathcal{F}_* \longrightarrow \mathcal{F}''_{*s} \longrightarrow 0,$$

for $s \in S$. This is actually a consequence of Lange's results [Ln], so we just introduce the argument needed to adapt them to the parabolic case.

Let π_S be the projection $X \times S \rightarrow S$ and $R^i \pi_{S*} \mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)$ be the higher direct image sheaves, for $i = 0, 1$. For $s \in S$, denote by

$$\tau_s^i : R^i \pi_{S*} \mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*) \otimes k(s) \rightarrow H^i(\mathcal{H}om(\mathcal{F}''_{*s}, \mathcal{F}'_{*s}))$$

the natural base change morphism. The condition that τ_s^i is an isomorphism for all points $s \in S$ will be shortened in $R^i \pi_{S*} \mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)$ *commutes with base change*.

Let $\mu : S' \rightarrow S$ be a morphism of schemes and denote by

$$E_*(S') = H^0(S', R^1 \pi_{S'*} \mathcal{H}om(\mu^* \mathcal{F}''_*, \mu^* \mathcal{F}'_*)).$$

Then E_* actually is a functor from the category of S -schemes to the category of sets. In fact, let $\nu : S'' \rightarrow S'$ be a morphism of schemes over S . This gives a map $E_*(S') \rightarrow E_*(S'')$ by composition of the natural map

$$H^0(S', R^1 \pi_{S'*} \mathcal{H}om(\mu^* \mathcal{F}''_*, \mu^* \mathcal{F}'_*)) \rightarrow H^0(S'', \nu^* R^1 \pi_{S'*} \mathcal{H}om(\mu^* \mathcal{F}''_*, \mu^* \mathcal{F}'_*))$$

and the morphism induced in cohomology by the base change morphism

$$\nu^* R^1 \pi_{S'*} \mathcal{H}om(\mu^* \mathcal{F}''_*, \mu^* \mathcal{F}'_*) \rightarrow R^1 \pi_{S''*} \nu^* \mathcal{H}om(\mu^* \mathcal{F}''_*, \mu^* \mathcal{F}'_*).$$

Since it is $\nu^* \mathcal{H}om(\mu^* \mathcal{F}''_*, \mu^* \mathcal{F}'_*) \cong \mathcal{H}om(\nu^* \mu^* \mathcal{F}''_*, \nu^* \mu^* \mathcal{F}'_*)$, this gives the morphism $E_*(S') \rightarrow E_*(S'')$.

Proposition 5.3. ([Ln], proposition 3.1) *Suppose that $R^i\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)$ commutes with base change, for $i = 0, 1$. Then the functor E_* is representable by the bundle associated with the locally free sheaf $R^1\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)^\vee$.*

Let $PE_*(S')$ denote the set of invertible quotients

$$R^1\pi_{S*}\mathcal{H}om(\mu^*\mathcal{F}''_*, \mu^*\mathcal{F}'_*)^\vee \rightarrow \mathcal{L}.$$

This defines a functor from the category of S -schemes to the category of sets.

Proposition 5.4. ([Ln], proposition 4.2) *Suppose that $R^i\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)$ commutes with base change, for $i = 0, 1$. Then the functor PE_* is representable by the projective bundle $P(R^1\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)^\vee)$.*

This result is applied in the proof of theorem 1.1 in the following way. Suppose that, for all $s \in S$, there is an isomorphism induced by base change

$$R^1\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*) \otimes k(s) \cong H^1(\mathcal{H}om(\mathcal{F}''_{*s}, \mathcal{F}'_{*s}))$$

and $H^0(\mathcal{H}om(\mathcal{F}''_{*s}, \mathcal{F}'_{*s})) = 0$. Then for $i = 0, 1$ the sheaves $R^i\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)$ commute with base change, the sheaf $R^1\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)$ is locally free over S and its fibre over a point s is isomorphic to $H^1(\mathcal{H}om(\mathcal{F}''_{*s}, \mathcal{F}'_{*s}))$. By proposition 5.4 the projective bundle associated with the sheaf $R^1\pi_{S*}\mathcal{H}om(\mathcal{F}''_*, \mathcal{F}'_*)^\vee$ parametrizes isomorphism classes of nonsplitting parabolic extensions of parabolic bundles of the family \mathcal{F}''_* by parabolic bundles of the family \mathcal{F}'_* .

5.3. Proof of theorem 1.1.

We first prove the theorem for generic weights of $\partial_0 W$: suppose that the weights $((d_j(p))_{p \in I}, k)$ do not lie on any Seshadri wall.

Consider the stratification of $\mathcal{E}_{r''}$ given by the quasi-parabolic invariants of the images of parabolic morphisms. Let $\mathcal{E}_{((n'_i(p))_{p \in I}, d'')}$ be the family of isomorphism classes of stable parabolic bundles F_* , such that there exists a morphism $\varphi_* : E_* \rightarrow F_*$ for which the vector bundle $\text{Im}(\varphi) = V$ has degree d'' and the induced parabolic structure on V' , the saturation of V in F , has multiplicities $((n'_i(p))_{p \in I})$. The parabolic morphism φ_* gives rise to a commutative diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & V & \longrightarrow & V'_* & \longrightarrow & \tau & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & F_* & \longrightarrow & G' & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & G_* & = & G & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}$$

and so the exact sequence in the second column is a nonsplitting parabolic extension. The bundle G_* has rank $\ell k - r''$, parabolic multiplicities

$$((\ell d_1(p) - n'_1(p), \dots, \ell d_{l_p}(p) - n'_{l_p}(p))_{p \in I})$$

and if $t = \deg(\tau)$, then $\deg(G) = \deg(F) - (d'' + t)$.

Let $\mathcal{V}'_{((n'_i(p))_{p \in I}), d'', t}$ be the family of isomorphism classes of parabolic bundles V'_* of degree $d'' + t$, multiplicities $((n'_i(p))_{p \in I})$, such that there exists a stable bundle F_* and a morphism $\varphi_* : E_* \rightarrow F_*$ for which $\text{Im}(\varphi)$ generates V' as a subbundle of F . Any such bundle is an extension of a torsion sheaf τ of degree t by a quotient bundle V of E and by inequalities 10 of remark 5.2, if the quotient morphism φ induces the parabolic structure V_* , then it is

$$\deg(V_*) \leq d'' + \frac{1}{k} \sum_{p \in I} \sum_{i > j} n'_i(p) d_j(p).$$

Denote by d''_* the right hand side of this inequality. This condition implies that the quotient $E \twoheadrightarrow \text{Im}(\varphi)$ is a point of a finite union of parabolic strata of the scheme $\text{Quot}_{r'', d''}(E)$, that we denote by $\mathbb{Q}_{((n'_i(p))_{p \in I}), d''}$. More explicitly, it is the union of those strata that correspond to functions s_* , for which $\int_{-1}^0 s_\alpha d\alpha \leq d''_*$. By the computation of theorem 3.6, its dimension is bounded by

$$(11) \quad \dim(\mathbb{Q}_{((n'_i(p))_{p \in I}), d''}) \leq r''(r - r'') + r(d''_* - d_{r'', *}),$$

where $d_{r'', *}$ is the minimal parabolic degree of a rank r'' quotient bundle of E .

Remark that in the generic case it is $\tau \cap I = \emptyset$ and then it is enough to consider those quotient morphisms of the stratum $\text{Quot}_{r'', s_*}^{par}(E_*)$, corresponding to the fixed multiplicities $((n'_i(p))_{p \in I})$. In any case, we draw the following estimate for the dimension of the family:

$$\dim(\mathcal{V}'_{((n'_i(p))_{p \in I}), d'', t}) \leq r''(r - r'') + r(d''_* - d_{r'', *}) + tr''.$$

Let $\mathcal{G}_{((n'_i(p))_{p \in I}), d'', t}$ be the family of isomorphism classes of parabolic bundles G_* of rank $\ell k - r''$, degree $\ell k(\mu(E_*) + g - 1) - (d'' + t)$ and multiplicities $((\ell d_i(p) - n'_i(p))_{p \in I})$, which are parabolic quotients of a stable bundle F_* by a bundle $V_* \in \mathcal{V}'_{((n'_i(p))_{p \in I}), d'', t}$. Consider the family of isomorphism classes of the underlying vector bundles and denote it by \mathcal{G} . This family is bounded. In fact, since any bundle G of \mathcal{G} is quotient of some parabolic stable bundle F_* , if we consider a rank n quotient bundle $G \twoheadrightarrow H$, there is a constant $h(n)$ such that $\mu(H) \geq h(n)$. This condition ensures the boundedness of \mathcal{G} . Thus there exists a scheme S and a vector bundle \mathcal{H} on $X \times S$ such that, for all G of the family \mathcal{G} there is an isomorphism $G \cong \mathcal{H}_s$, for some $s \in S$. By [BP-Gr-Ne], lemma 4.1, we can suppose that the generic bundle of the family \mathcal{G} is semistable, *i.e.* $\dim(\mathcal{G}) \leq (\ell k - r'')^2(g - 1) + 1$. For all $p \in I$, let $\iota_p : \{p\} \times S \hookrightarrow X \times S$ denote the inclusion morphism and consider the bundle in flag varieties

$$\mathcal{F}_p = \mathcal{F}\text{lag}_{(\ell d_1(p) - n'_1(p), \dots, \ell d_{l_p}(p) - n'_{l_p}(p))}(\iota_p^* \mathcal{H}).$$

Let \mathcal{F} denote the fibred product over $X \times S$ of the bundles \mathcal{F}_p . Recall that its dimension is given by

$$\dim(\mathcal{F}) = \dim(S) + \sum_{p \in I} \sum_{i > j} (\ell d_i(p) - n'_i(p)) (\ell d_j(p) - n'_j(p)).$$

This family parametrizes quasi-parabolic bundles, whose underlying vector bundle is isomorphic to \mathcal{H}_s , for some $s \in S$. Thus $\mathcal{G}_{((n'_i(p))_{p \in I}), d'', t}$ is a bounded family and moreover it is

$$\dim(\mathcal{G}_{((n'_i(p))_{p \in I}), d'', t}) \leq (\ell k - r'')^2 (g - 1) + 1 + \sum_{p \in I} \sum_{i > j} (\ell d_i(p) - n'_i(p)) (\ell d_j(p) - n'_j(p)).$$

Let $\mathcal{E}_{((n'_i(p))_{p \in I}), d'', t}$ denote the family of isomorphism classes of stable parabolic bundles F_* , which are parabolic extensions of a bundle $G_* \in \mathcal{G}_{((n'_i(p))_{p \in I}), d'', t}$ by a bundle $V'_* \in \mathcal{V}'_{((n'_i(p))_{p \in I}), d'', t}$.

Lemma 5.5. *Let F_* be a stable parabolic bundle and*

$$0 \longrightarrow F'_* \xrightarrow{i_*} F_* \xrightarrow{p_*} F''_* \longrightarrow 0$$

a parabolic extension. Then $\text{Hom}(F''_, F'_*) = 0$.*

Proof: If there were a nonzero parabolic morphism $\varphi_* : F''_* \rightarrow F'_*$, there would be an endomorphism of F_* , that is $i_* \varphi_* p_* : F_* \rightarrow F_*$, which is not a multiple of the identity. \square

From this lemma it follows that $h^0(\mathcal{H}om(G_*, V'_*)) = 0$ and so the dimension of $H^1(\mathcal{H}om(G_*, V'_*)) \cong \text{Ext}^1(G_*, V'_*)$ is constant for all V'_* of $\mathcal{V}'_{((n'_i(p))_{p \in I}), d'', t}$ and G_* of $\mathcal{G}_{((n'_i(p))_{p \in I}), d'', t}$. Therefore we can compute the dimension of the first cohomology group as the opposite of the Euler characteristic:

$$\begin{aligned} & \dim(\text{Ext}^1(G_*, V'_*)) \\ &= \sum_{p \in I} \sum_{i > j} (\ell d_i(p) - n'_i(p)) n'_j(p) - (\ell k - r'')(d'' + t) \\ & \quad + r''(\ell k(g - 1 + \mu(E_*)) - (d'' + t)) + r''(\ell k - r'')(g - 1) \\ &= -\ell k(d'' + t) + 2r''\ell k(g - 1) - r''^2(g - 1) + r''\ell k\mu(E_*) \\ & \quad + \sum_{p \in I} \sum_{i > j} (\ell d_i(p) - n'_i(p)) n'_j(p). \end{aligned}$$

Proposition 5.4 then gives the following bound for the dimension of the family of extensions

$$\begin{aligned} & \dim(\mathcal{E}_{((n'_i(p))_{p \in I}), d'', t}) \\ & \leq \dim(\mathcal{V}'_{((n'_i(p))_{p \in I}), d'', t}) + \dim(\mathcal{G}_{((n'_i(p))_{p \in I}), d'', t}) + h^1(\mathcal{H}om(G_*, V'_*)) - 1. \end{aligned}$$

The computation then goes as follows:

$$\begin{aligned}
& \dim(\mathcal{E}_{((n'_i(p))_{p \in I}, d'', t)}) \\
& \leq \dim(\mathbb{Q}_{((n'_i(p))_{p \in I}, d'')}) + r''t + (\ell k - r'')^2(g-1) \\
& \quad + \sum_{p \in I} \sum_{i > j} (\ell d_i(p) - n'_i(p))(\ell d_j(p) - n'_j(p)) + 1 - \ell k(d'' + t) + r'' \ell k \mu(E_*) \\
& \quad + r''(2\ell k - r'')(g-1) + \sum_{p \in I} \sum_{i > j} (\ell d_i(p) - n'_i(p))n'_j(p) - 1 \\
& = (\ell k)^2(g-1) + \dim(\mathbb{Q}_{((n'_i(p))_{p \in I}, d'')}) + \sum_{p \in I} \sum_{i > j} \ell d_i(p) \ell d_j(p) \\
& \quad - \sum_{p \in I} \sum_{i > j} n'_i(p) \ell d_j(p) - \ell k d'' + t(r'' - \ell k) + r'' \ell k \mu(E_*) \\
& = (\ell k)^2(g-1) + \sum_{p \in I} d_{\ell d_1(p), \dots, \ell d_{i_p}(p)} + t(r'' - \ell k) + \dim(\mathbb{Q}_{((n'_i(p))_{p \in I}, d'')}) \\
& \quad - \ell(kd'' + \sum_{p \in I} \sum_{i > j} n'_i(p) d_j(p) - \frac{r''}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p)).
\end{aligned}$$

The right hand side of the inequality should be read as

$$\begin{aligned}
& \dim(M_\ell^{p a r}) - 1 + t(r'' - \ell k) + \dim(\mathbb{Q}_{((n'_i(p))_{p \in I}, d'')}) \\
& \quad - \ell(kd'' + \sum_{p \in I} \sum_{i > j} n'_i(p) d_j(p) - \frac{r''}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p)).
\end{aligned}$$

By assumption it is $\ell k \geq r \geq r''$ and $t \geq 0$, so to prove the theorem it is enough to show that

$$\begin{aligned}
& \dim(\mathbb{Q}_{((n'_i(p))_{p \in I}, d'')}) \leq \\
& \quad \ell(kd'' + \sum_{p \in I} \sum_{i > j} n'_i(p) d_j(p) - \frac{r''}{r} \sum_{p \in I} \sum_{i > j} n_i(p) d_j(p)).
\end{aligned}$$

We can rewrite the right hand side as $\ell k(d''_* - r'' \mu(E_*))$. Thus, by inequality 11 it will be enough to show that

$$r''(r - r'') + r(d''_* - d_{r'', *}) \leq \ell k(d''_* - r'' \mu(E_*)),$$

that we can rewrite as

$$(12) \quad r''(r - r'') \leq (\ell k - r)(d''_* - d_{r'', *}) + \ell k(d_{r'', *} - r'' \mu(E_*)).$$

By assumption it is $\ell \geq \frac{r}{k}$ and by remark 5.2 d''_* is greater than or equal to the minimal parabolic degree $d_{r'', *}$. So in order to get inequality 12 it is enough to show that

$$r''(r - r'') \leq \ell k(d_{r'', *} - r'' \mu(E_*)).$$

This inequality is trivial, when $r'' = r$, since in this case both sides are equal to zero.

Remark 5.6. Suppose r'' is strictly less than r . Then it is $\ell \leq \ell k(d_{r'',*} - r''\mu(E_*))$. In fact, let V be a quotient bundle of rank r'' and minimal parabolic degree $d_{r'',*}$. The level k of the parabolic structure is such that $k\mu(E_*) \in \mathbf{Z}$ and moreover $kr''\mu(V_*) = k \deg(V_*)$ is an integer as well. Hence the difference $kr''(\mu(V_*) - \mu(E_*))$ is an integer, which is strictly positive since E_* is stable. Then we draw the inequality

$$\ell \leq \ell kr''(\mu(V_*) - \mu(E_*)) = \ell k(d_{r'',*} - r''\mu(E_*)).$$

By this remark inequality 12 for the nontrivial case $r'' < r$ follows from the assumption $\ell \geq r''(r - r'')$. This finishes the proof of theorem 1.1 for generic weights.

We are left with the case in which the weights $((d_j(p))_{p \in I}, k)$ of the parabolic structure are on a Seshadri wall and the bundle E_* is strictly semistable. Let $E_* = E_{0*} \supset E_{1*} \supset \cdots \supset E_{n*} \supset 0$ denote a Jordan-Hölder filtration of E_* , that is each quotient E_{h*}/E_{h+1*} is a stable bundle of parabolic slope $\mu(E_*)$. By the previous computation, for each h there is an open subscheme \mathcal{U}_h of the moduli space $M_\ell^{\prime par}$ such that, for all stable bundle F_* whose isomorphism class is in \mathcal{U}_h , it is $\text{Hom}(E_{h*}/E_{h+1*}, F_*) = 0$. Since the moduli space $M_\ell^{\prime par}$ is irreducible, the open subscheme $\mathcal{U} = \bigcap_{h=1}^l \mathcal{U}_h$ is nonempty and by definition, for all stable bundle F_* whose isomorphism class is in \mathcal{U} it is $\text{Hom}(E_*, F_*) = 0$. This finishes the proof of the theorem.

This bound for the order of base point freeness does not depend on the degree $|I|$ of the parabolic divisor and extends the result of Popa and Roth for the classical case as well as the result of Pauly for rank 2 parabolic bundles with generic parabolic divisor of small degree.

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