Christian Peskine

The k-secant Lemma

We report on a joint work with Laurent Gruson. Let X be a smooth variety imbedded in $\mathbb{P}_N$. For $k = k_1 + ... + k_r$, we study the Hilbert-Boardman Scheme $HB_{k_1,...,k_r}(X)$ of X. It parametrizes the finite linear subschemes $Z \subset X$ supported in r points $x_1, ..., x_r$ (eventually coinciding) and such that $k = d^0(Z)$ and $d^0_{x_i}(Z) = k_i$. We consider the incidence variety $I_{HB} = \{(Z, y) \in HB_{k_1,...,k_r}(X) \times \mathbb{P}_N$ such that $Z \cup \{y\}$ is on a line$\}$, and we prove the k-secant lemma: the general fiber of the second projection $(I_{HB} \to \mathbb{P}_N)$ is smooth of dimension $N - 1 - kc + r$, where c is the codimension of X in $\mathbb{P}_N$ (the empty set has all dimensions).

As a special case (when $r = 1$ and $k = k_1$), we recover a celebrated theorem of Mather : higher polar varieties (for a general point) of a smooth variety are smooth.

Ziv Ran's famous $(n + 2)$-secant lemma, as well as a recent generalisation of this lemma by R. Beheshti and D. Eisenbud, can also be recovered as a consequence of our result.

As an obvious corollary of the k-secant lemma, we find that the general projection X_1 of X to $\mathbb{P}_{N-1}$ has generic singularities. In particular, the subvariety $X_k \subset X_1$ formed by points of multiplicity $\geq k$ has codimension $k(c - 1)$ in $\mathbb{P}_{N-1}$, and is smooth outside X_{k+1}. To be more precise, the subvariety $X_{\{k_1,...,k_r\}} \subset X_1$ formed by points whose fiber in X has support in r points $\{x_1, ..., x_r\}$ (eventually coinciding) and multiplicity $\geq k_i$ at x_i, has codimension $kc - r$ in $\mathbb{P}_{N-1}$, and is smooth outside $X_{\{k_1,...,k_r,1\}} = X_{\{k_1,...,k_r\}} \cap X_{k+1}$.

As a consequence of this corollary, we prove a vanishing theorem for codimension 2 smooth varieties, that we conjectured in Trento twenty years ago: $H^p(J_X(q)) = 0$ for $p + q < N - 2$ and $p < N - 1$. Finally, with the help of Fyodor Zak, we conjecture a generalization of this result for smooth projective varieties of any codimension in a projective complex space, to be compared with Barth and Zak classical vanishing theorems.