LECTURE 1: CLASSICAL METHODS IN RESTRICTION THEORY

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1. Basic restriction theory

The purpose of these notes is to describe some exciting recent work of Bourgain and Demeter [4] in Fourier restriction theory. Here we will focus on restriction to a compact piece of the parabola, defining
\[ P^n := \{(x, |x|^2) : x \in [-1,1]^{n-1}\} \]
and letting \( \sigma \) denote the surface measure supported on \( P^n \) given by
\[ \int f \, d\sigma := \int_{[-1,1]^{n-1}} f(x, |x|^2) \, dx. \]

We begin by recalling the following fundamental conjecture.

Conjecture 1 (Fourier Restriction Conjecture). The inequality
\[ \|\hat{f}\|_{L^q(P^n, \sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \]
holds whenever \( p > 2n/(n-1) \) and \( q \leq (n-1)p/(n+1) \).

Throughout these notes, the dependence of constants on the dimension \( n \) and any Lebesgue exponents will be suppressed in the \( \lesssim \) notation.

As always, the conjecture requires (1) to hold whenever \( f \) is assumed to belong to some suitable dense class of functions, \( \hat{f} \) denotes the Fourier transform of \( f \) and \( \hat{f}|_{P^n} \) its restriction to \( P^n \). Thus, the problem is to determine the mapping properties of the restriction operator \( R: f \mapsto \hat{f}|_{P^n} \). This appears to be an extremely difficult question and despite a vast amount of work by many pre-eminent mathematicians, only the \( n = 2 \) case is known in full (due to Fefferman and Zygmund). There is a complicated array of partial results in higher dimensions (i.e. with restricted ranges of \( p \) and \( q \)) which we will not survey here.

It is often convenient to work with the adjoint extension operator leading to an equivalent formulation of Conjecture 1.

Conjecture 2 (Fourier Restriction Conjecture (Adjoint Form)). The inequality
\[ \|g\|_{L^{p'}(\mathbb{R}^n)^*} \lesssim \|g\|_{L^{q'}(P^n, \sigma)^*} \]
holds whenever \( p' > 2n/(n-1) \) and \( q \leq (n-1)p'/(n+1) \).

The restriction conjecture has a rich history and is intimately connected with numerous other important questions in harmonic analysis and beyond. However, here such details, fascinating as they are, are not our main concern and we are content to consider Conjectures 1 and 2 without further motivation.

\[ \text{1To be precise, the surface measure is a weighted version of \( \sigma \); the weight is } \approx 1 \text{ everywhere on } P^n \text{ and therefore innocuous as far as we are concerned and so we choose to ignore it. Further, we remark that } \sigma \text{, though not precisely the surface measure, has a geometric interpretation as the affine surface measure on } P^n. \text{ This choice of weight has some special significance in restriction theory but we will not focus at all on the “affine perspective” here.} \]

\[ \text{2The interested reader is directed to the excellent and accessible survey article [12] and course notes [10] for background information.} \]
There is always a trivial range of estimates for which (1). Indeed, if \( f \in L^1(\mathbb{R}^n) \), then by Riemann-Lebesgue \( \hat{f} \) is a continuous function satisfying
\[
\|\hat{f}\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R}^n)}.
\]
From these facts one immediately deduces the conjectures always hold in the restricted range given by \( p = 1, 1 \leq q \leq \infty \).

**Remark 3.** One geometric property of the parabola is fundamental to the problem: \( P^{n-1} \) has everywhere non-vanishing Gaussian curvature.\(^3\) It is natural to formulate a general conjecture with the parabola replaced by any (compact) hypersurface \( S \) satisfying this property (an important alternative example of such a hypersurface being the sphere). However, for simplicity of exposition, we will adhere to the concrete situation presented above. If the curvature of the surface is allowed to vanish, then the full range of estimates given in the statement of the conjectures is not possible. In the extreme case, one can show relatively easily that the trivial range of estimates discussed in the previous remark is best possible if and only if \( S \) contains a point at which the Gaussian curvature vanishes to infinite order.

The range of exponents in Conjecture 1 is suggested by testing the inequality on some simple examples:

- The condition \( p' > 2n/(n - 1) \) is due to the well-known fact that \( (d\sigma)^\sim(\xi) \) has a decay rate of the order of \( |\xi|^{-(n-1)/2} \) as \( |\xi| \to \infty \) (which itself is a consequence of the curvature properties discussed above).
- We mention in particular that the condition \( q \leq (n-1)p'/(n+1) \) arises from the Knapp example. Let \( R \gg 1 \) and consider the small rectangle
  \[
  D := [-R^{-1/2}, R^{-1/2}]^{n-1} \times [-R^{-1}, R^{-1}].
  \]
  The intersection of \( D \) with \( P^{n-1} \) is a \( R^{-1/2} \times \cdots \times R^{-1/2} \) cap on the surface, we let \( g \) denote its characteristic function so that
  \[
  \|g\|_{L^{p'}(P^{n-1}, \sigma)} \sim R^{-(n-1)/2q'}.
  \]
  By the uncertainty principle one expects \( (gd\sigma)^\sim \) to be concentrated on the cone generated by the normals to the surface along the cap. In particular, it is reasonable to suppose that this function should be large on the dual rectangle
  \[
  D^* := [-R^{1/2}, R^{1/2}]^{n-1} \times [-R, R].
  \]
  Indeed, it is easy to see \( |(gd\sigma)^\sim(x)| \gtrsim R^{-(n-1)/2} \chi_{D^*}(x) \) leading to the estimate
  \[
  \|g(d\sigma)^\sim\|_{L^{p'}(\mathbb{R}^n)} \gtrsim R^{-(n-1)/2}|D^*|^{1/p'} = R^{-(n-1)/2 + (n+1)/2p'}.
  \]
  Comparing (3) and (4) gives the desired condition on the exponents.
  Notice that one could take \( D^* \) to be any \( R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1} \) rectangle which is centred “tangentially” at some point on the parabola (and therefore intersects \( P^{n-1} \) on a \( R^{-1/2} \times \cdots \times R^{-1/2} \) cap). In this case, the dual rectangle \( D^* \) is centred at the origin and has the same dimensions as before but is now orientated in the normal direction to the centre of the cap.

One important partial result on the restriction conjecture is the Stein-Tomas theorem, which provides a sharp \( L^2 \)-based estimate.

\(^3\)The fact \( P^{n-1} \) is compact or, moreover, that \( \sigma \) is a finite measure is also significant, but one may develop a meaningful restriction theory for non-compact surfaces provided a certain amount of symmetry is present in the problem.
\textbf{Theorem 4 (Stein-Tomas restriction theorem).} For $1 \leq p \leq 2(n+1)/(n+3)$ we have
\[
\left| \hat{f} \right|_{L^p(S^{n-1}, \sigma)} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^n)}.
\]

Here we present a simple argument due to Carbery which gives the full range of inequalities except for the $p = 2(n+1)/(n+3)$ endpoint, where a restricted-type estimate is obtained. This weakened version of Theorem 4 is a consequence of the following lemma and interpolation (with the trivial estimates).

\textbf{Lemma 5 (Almost sharp Stein-Tomas).} For any Borel set $E \subset \mathbb{R}^n$, we have
\[
\int_{\mathbb{R}^n} \left| \hat{\chi}_E(\xi) \right|^2 d\sigma(\xi) \lesssim \left\| \chi_E \right\|_{L^2(n+1)/(n+3)}(\mathbb{R}^n).
\]

\textbf{Proof.} Let $T$ denote the convolution operator $Tf := f \ast \tilde{\mu}$ and fix a Borel set $E \subset \mathbb{R}^n$. Observe, by duality,
\[
\int_{\mathbb{R}^n} \left| \hat{\chi}_E(\xi) \right|^2 d\sigma(\xi) = \int_{\mathbb{R}^n} \chi_E(x) \hat{T}\chi_E(x) dx
\]

We proceed by estimating the convolution operator.\footnote{This trick of reducing the problem of estimating the restriction operator to estimating the convolution operator is an example of the so-called TTT* method. A similar argument can be applied to obtain Strichartz estimates for solutions to the Schrödinger equation in $\mathbb{R}^n$, as discussed later in these notes.} Fix a radially decreasing Schwartz function $\varphi$ satisfying $\varphi(x) = 1$ for $x \in B(0,1)$ and $\text{supp}(\varphi) \subset B(0,2)$ and for $r > 0$ define $\varphi_r(x) := \varphi(r^{-1}x)$. Decompose the measure $\sigma$ by writing $\sigma = \sigma_1 + \sigma_2$ where $\sigma_1(-x) = \tilde{\sigma}(x)\varphi_r(x)$ and $\sigma_2(-x) = \tilde{\sigma}(x)(1 - \varphi_r(x))$ for some appropriate value of $r > 0$ to be chosen later. Thus, $T = T_1 + T_2$ where $T_j := f \ast \tilde{\sigma}_j$ for $j = 1, 2$ and it suffices to show
\[
|E|^{1/2} \left\| T_1 \chi_E \right\|_{L^2(\mathbb{R}^n)} + \left\| T_2 \chi_E \right\|_{L^\infty(\mathbb{R}^n)} \lesssim \left\| \chi_E \right\|_{L^2(n+1)/(n+3)}^2(\mathbb{R}^n).
\]

Observe $\sigma_1(\xi) = \hat{\varphi}_r * \sigma(\xi)$ and so
\[
\sigma_1(\xi) = \int_{B(\xi, 1/r)} \hat{\varphi}_r(\xi - \eta) d\sigma(\eta) + \sum_{k=1}^{\infty} \int_{B(\xi, 2^k/r) \setminus B(\xi, 2^{k-1}/r)} \hat{\varphi}_r(\xi - \eta) d\sigma(\eta)
\]

\[
= I + II
\]

Using the simple estimate $\left\| \hat{\varphi}_r \right\|_{L^\infty(\mathbb{R}^n)} \lesssim |B(0, 2r)|$, one may deduce
\[
|I| \lesssim |B(0, 2r)| |\sigma(B(\xi, 1/r) \cap P^{n-1})| \lesssim r,
\]

where the latter inequality is due to the dimensionality of $P^{n-1}$. Furthermore, the rapid decay of $\hat{\varphi}$ implies $|\hat{\varphi}_r(\xi - \eta)| \lesssim 2^{-(k-1)n} r^n$ whenever $\eta \notin B(\xi, 2^{k-1}/r)$ and so
\[
|II| \lesssim \left( \sum_{k=1}^{\infty} 2^{-(k-1)n} |\sigma(B(\xi, 2^k/r) \cap P^{n-1})| \right) r^n \lesssim \left( \sum_{k=1}^{\infty} 2^{-k} \right) r.
\]

Combining these observations
\[
\left\| \sigma_1 \right\|_{L^\infty(\mathbb{R}^n)} \lesssim r \text{ and so}
\]

\[
\left\| T_1 \chi_E \right\|_{L^2(\mathbb{R}^n)} = \left\| \sigma_1 \hat{\chi}_E \right\|_{L^2(\mathbb{R}^n)} \lesssim r |E|^{1/2}.
\]

On the other hand, stationary phase calculations show the measure $\sigma$ satisfies the Fourier decay estimate
\[
|\tilde{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-(n-1)/2}
\]

and, since $\text{supp}(1 - \varphi_r) \subset \mathbb{R}^n \setminus B(0, r)$, it follows $\left\| \tilde{\sigma}_2 \right\|_{L^\infty(\mathbb{R}^n)} \lesssim r^{-(n-1)/2}$ and hence
\[
\left\| T_2 \chi_E \right\|_{L^\infty(\mathbb{R}^n)} = \left\| \tilde{\sigma}_2 \right\|_{L^\infty(\mathbb{R}^n)} \left\| \chi_E \right\|_{L^1(\mathbb{R}^n)} \lesssim r^{-(n-1)/2} |E|.
\]
Thus, combining these observations and choosing \( r \) so that \( r^{1+(n-1)/2} \sim |E| \), the desired estimate (6) follows.

□

Remarks. 1) Inequalities of the form (5) were first considered by Stein in unpublished work dating back to the late 1960s; sharp estimates were later established by Stein and Tomas. More precisely, Stein obtained the sharp result by using analytic interpolation techniques whilst Tomas provided a much simpler argument which gave estimates only in the restricted range \( 1 \leq p < 2(n+1)/(n+3) \), missing the endpoint. It was observed by Carbery that Tomas’ proof can be adapted to give a restricted-type inequality when \( p = 2(n+1)/(n-3) \) (this is the argument presented above). More recently, Bak and Seeger developed these methods to obtain the full range of strong-type estimates and thereby gave an alternative and, it transpires, more robust proof of Stein’s theorem.

2) The argument presented above is quite general and relies only on the dimensionality and Fourier decay of the measure \( \sigma \). In particular, Mitsis and Mockenhaupt observed one may use these methods to formulate and prove Stein-Tomas-type theorems for general Borel measures \( \mu \) on \( \mathbb{R}^n \) which satisfy for some \( 0 < a < n \) and \( b \leq a \):\(^5\)

- The dimensionality condition
  \[ \mu(B(\xi, r)) \lesssim r^a \quad \text{for all } \xi \in \mathbb{R}; \]

- The Fourier decay condition
  \[ |\hat{\mu}(x)| \lesssim r^{-b/2} \quad \text{for all } -x \notin B(0, r). \]

One interesting consequence of this is that it allows one to develop a non-trivial Fourier restriction theory for subsets of \( \mathbb{R} \).

3) Moreover, in unpublished work Wright observed that one may push these methods further and apply them in a general locally compact abelian group \( G \), provided \( G \) admits a basic form of Littlewood-Paley theory. One particularly interesting example is given by \( G := [Z/pZ]^n \); in this setting one may develop a discretised analogue of (Euclidean) restriction theory.

4) An argument of Bak and Seeger can be applied in these abstract settings to strengthen all the results to strong-type estimates at the relevant endpoints.

2. Restriction and Kakeya

The Knapp example introduced in the previous section is central to much of the following discussion. Note that, necessarily, the restriction estimate (2) “just fails” for the exponents \( p' = q' = 2n/(n-1) \) (albeit the failure is rather dramatic [1]). In particular, one can show using Hölder’s inequality that if Conjecture 1 is true, then for all \( \epsilon > 0 \), the estimate

\[ \|g\sigma\|_{L^{2n/(n-1)}(B(0,CR))} \lesssim R^{\epsilon} \|g\|_{L^{2n/(n-1)}(P^{n-1},\sigma)} \]

holds for all \( R \gg 1 \). Such “\( \epsilon \)-loss” in \( R \) will be a regular feature of our analysis and it is useful to introduce the following notation: if \( X,Y \) are non-negative real numbers then \( X \lesssim Y \) or \( Y \gg X \) is taken to mean that for all \( \epsilon > 0 \) we have \( X \lesssim, R^\epsilon Y \) for all \( R \gg 1 \). Hence, (8) can be concisely expressed as

\[ \|g\sigma\|_{L^{2n/(n-1)}(B(0,CR))} \lesssim \|g\|_{L^{2n/(n-1)}(P^{n-1},\sigma)}. \]

Fix \( R \gg 1 \) and consider many Knapp examples placed around the parabola; that is, take \( g = \sum_n \chi_n \) to be the sum of characteristic functions of many disjoint

\(^5\)These restrictions on the parameters \( a,b \) are natural; see, for instance, [13].
$R^{-1/2} \times \cdots \times R^{-1/2}$ caps on $P^{n-1}$. Letting $\Omega \subseteq S^{n-1}$ denote the set normal directions to these caps, we have

$$\|g\|_{L^{2n/(n-1)}(P^{n-1})} \sim \left( R^{-(n-1)/2} \# \Omega \right)^{(n-1)/2n}.$$  \hfill (9)

On the other hand, our heuristics tell us $(\chi_\omega d\sigma)^-$ is large on some $\sim R^{1/2} \times \cdots \times R^{1/2} \times R$ rectangle $T_\omega$ which is centred at the origin and orientated in the direction $\omega$. Thus, heuristically,

$$(gd\sigma)^-(x) \sim R^{-(n-1)/2} \sum_{\omega \in \Omega} e^{2\pi i \xi_\omega \cdot x} \chi_{T_\omega}(x)$$  \hfill (10)

for some collection of frequencies $\xi_\omega$. By modulating the summands of $g$ one may replace each $T_\omega$ in (10) with any translate of itself whilst maintaining (9). We stipulate that the tubes are contained in $B(0, CR)$, but otherwise arrange them in an arbitrary fashion.

There is some cancellation arising from the exponentials in (10) and we manipulate this using a standard randomisation argument. In very rough and non-rigorous terms, randomisation allows us to model the cancellation by the estimate

$$|(gd\sigma)^-(x)| \lesssim R^{-(n-1)/2} \left( \sum_{\omega \in \Omega} \chi_{T_\omega}(x) \right)^{1/2}.$$  \hfill (11)

If no cancellation was present, then $|(gd\sigma)^-(x)|$ would roughly be the product of $R^{-(n-1)/2}$ and an $\ell^2$ sum of the $\chi_{T_\omega}$; the presence of the modulations allows us to knock this down to a (smaller) $\ell^2$ sum. In practice one works with the expected values of a randomised version of $g$ and shows (11) holds “on average”.

To make the above discussion precise, let $\epsilon_\kappa = \pm 1$ denote a sequence of independent, identically distributed random signs which take the values $+1$ and $-1$ with equal probability $1/2$. We assign to each cap $\kappa$ a (unique) random sign $\epsilon_\kappa$ and redefine $g$ by taking

$$g(\xi) = \sum_{\kappa} \epsilon_\kappa e^{2\pi i x_\kappa \cdot \xi} \chi_\kappa(\xi)$$

for some choice of $x_\kappa \in \mathbb{R}^n$, so that

$$(gd\sigma)^-(x) = \sum_{\kappa} \epsilon_\kappa (\chi_\kappa d\sigma)^-(x - x_\kappa).$$

The randomisation can be exploited by appealing to Khinchin’s inequality. This states that for all $0 < p < \infty$, if $\epsilon_k$ are as above,

$$\mathbb{E} \left[ \left| \sum_{k=1}^m \epsilon_k a_k \right|^p \right]^{1/p} \sim \left( \sum_{k=1}^m |a_k|^2 \right)^{1/2}$$

for any sequence $a_1, \ldots, a_m \in \mathbb{C}$; details of the proof can be found in [9]. In the present situation Khinchin’s inequality implies

$$\mathbb{E} \left[ \left| \left( gd\sigma \right)^-(x) \right|^{2n/(n-1)} \right]^{(n-1)/2n} \sim \left( \sum_{\kappa} \left| (\chi_\kappa d\sigma)^-(x - x_\kappa) \right|^2 \right)^{1/2} \lesssim R^{-(n-1)/2} \left( \sum_{\omega \in \Omega} \chi_{T_\omega}(x) \right)^{1/2}$$

for almost every $x \in \mathbb{R}^n$, which is the rigorous version (11). Thus, taking $L^p$-norms and applying Fubini’s theorem,

$$\mathbb{E} \left[ \left\| (gd\sigma)^- \right\|_{L^p(\mathbb{R}^n)} \right] \gtrsim R^{-(n-1)/2} \left\| \sum_{\omega \in \Omega} \chi_{T_\omega} \right\|^{1/2}_{L^{n/(n-1)}(\mathbb{R}^n)}$$  \hfill (12)
On the other hand, the value of $|g|$ is independent of the outcome of the $\epsilon_k$ and thus
\[
E[|g|_{L^{2n/(n-1)}(\mathbb{P}^{n-1})}] \sim (R^{-(n-1)/2} \#\Omega)^{(n-1)/2n}.
\]
Combining (8), (12) and (13) it follows
\[
\left| \sum_{\omega \in \Omega} \chi_{T_\omega} \right|_{L^{n/(n-1)}(\mathbb{R}^n)} \lesssim R^{n-1-(n-1)/2n} \#\Omega^{(n-1)/n}.
\]
Thus, we conclude that the restriction conjecture, if true, would imply the following result concerning the size of a union of distinctly-orientated rectangles.

**Conjecture 6** (Kakeya Maximal Conjecture). Let $\Omega \subseteq S^{n-1}$ be a maximal set of $R^{-1/2}$-separated directions and $\{T_\omega\}_{\omega \in \Omega}$ a collection of $R \times R^{1/2} \times \cdots \times R^{1/2}$ rectangles where $T_\omega$ is orientated in the direction of $\omega$. Then the inequality
\[
\left| \sum_{\omega \in \Omega} \chi_{T_\omega} \right|_{L^{n/(n-1)}(\mathbb{R}^n)} \lesssim \left( \sum_{\omega \in \Omega} |T_\omega| \right)^{(n-1)/n}
\]
holds.

**Remarks.**
a) In much of the literature the above conjecture is stated in an equivalent, rescaled form involving $1 \times R^{-1} \times \cdots \times R^{-1}$ rectangles. The scale used here is suited to the geometry restriction problem.
b) If the rectangles $T_\omega$ were mutually disjoint, then
\[
\left| \sum_{\omega \in \Omega} \chi_{T_\omega} \right|_{L^{n/(n-1)}(\mathbb{R}^n)} = \left( \sum_{\omega \in \Omega} |T_\omega| \right)^{(n-1)/n}
\]
and so (14) can be interpreted as a quantifying the principle that rectangles which point in distinct directions must have small intersection (or, in other words, they must be “essentially disjoint”).

The Kakeya conjecture is a major open problem in geometric measure theory which is closely connected to many classical problems in Fourier analysis. A simple and elegant argument of Córdoba can be used to establish the two dimensional case\(^6\) but in all higher dimensions only partial results are known.\(^7\)

### 3. Local restriction theory

The restriction conjecture implies the Kakeya conjecture and one is tempted to ask to what extent, if any, the converse holds. Since oscillation clearly plays a substantial rôle in the former problem whilst the latter is a pure size estimate (i.e. there is no oscillation present), it seems infeasible that Conjecture 1 should be a consequence of Kakeya alone. One method of capturing the cancellation present in Conjecture 1 is to attempt to interpose a certain kind of square function;\(^8\) we’ll see that if one assumes the necessary square function estimate holds, then the restriction conjecture follows from the Kakeya estimate (14). Before introducing these square functions, however, it will be useful to develop local restriction theory.

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\(^6\)Alternatively, Conjecture 6 can be proved for $n = 2$ using the (known) $n = 2$ case of the restriction conjecture via the method outlined above.

\(^7\)For further information, the surveys of Wolff [13] and Katz and Tao [8] provide an excellent introduction to the Kakeya conjecture.

\(^8\)This is similar in spirit to the estimate (11) in the proof of Restriction $\implies$ Kakeya, the key difference being that before we were constructing a specific function $g$ and therefore had the useful technique of randomisation at our disposal. Since now we want to prove restriction estimates for general $g$, it does not make sense to assume $g$ is randomised.
Rather than considering the restriction conjecture per se, most of the time we will focus on establishing ostensibly weaker local estimates. In particular, we consider inequalities of the form
\[ \| (g \partial x) \|_{L^{2n/(n-1)}(B_R)} \lesssim R^\alpha \| g \|_{L^{2n/(n-1)}(P^n-1)} \]  
(15)
for \( R \gg 1 \), where \( B_R \subset \mathbb{R}^n \) is an arbitrary ball of radius \( R \) (by the translation invariance of the problem, the choice of centre of the ball is irrelevant). It is clear that the local estimate (15) holds with \( \alpha = 0 \) if and only if the global restriction inequality (2) is true. Furthermore, it is easy to see (15) always holds for \( \alpha = n/p' \) and (15) becomes progressively harder to prove as \( \alpha \) decreases: the problem becomes to try to push down the value of \( \alpha \).

We have already had a glimpse of the utility of such local estimates in the proof of Restriction \( \implies \) Kakeya where it was noted that for any \( \epsilon > 0 \) the restriction conjecture implies the local estimate
\[ \| (g \partial x) \|_{L^{2n/(n-1)}(B_R)} \lesssim \epsilon R^\alpha \| g \|_{L^{2n/(n-1)}(P^n-1)} \].  
(16)
Indeed, one of the many advantages of this localised set-up is that it now makes sense to consider restriction estimates at the endpoint \( p' = q' = 2n/(n-1) \). More importantly, Tao [11] showed that (16) with its \( \epsilon \)-loss in the \( R \)-exponent can be converted into a global estimate with a loss in \( p \) (i.e. we move away from the endpoint); moreover, if the \( \epsilon \)-loss can be made arbitrarily small, then one may obtain global estimates arbitrarily close to the endpoint. From this one can then obtain the full range of estimates for the restriction conjecture via interpolation with the trivial range and Hölder’s inequality. Consequently, Conjecture 1 is in fact equivalent to showing (16) holds for arbitrarily small \( \epsilon \).

Thus, the restriction conjecture can be reformulated as:

**Conjecture 7** (Restriction conjecture, local form). The inequality
\[ \| (g \partial x) \|_{L^{2n/(n-1)}(B_R)} \lesssim \| g \|_{L^{2n/(n-1)}(P^n-1)} \]
holds for a suitable class of functions \( g \) on \( P^n-1 \).

Henceforth we will concentrate on proving localised restriction estimates, taking advantage of many phenomena which do not arise in the global setting. First of all, we observe that localising to scale \( R \) in the spatial variable should impact upon the situation on the frequency side via the uncertainty principle and one expects localisation to scale

**Definition** \((L^p\text{-averages})\). Let \( 1 \leq p < \infty \).

i) If \( \Lambda \) is a finite set and \( f : \Lambda \to \mathbb{C} \) is any function, then
\[ \| f \|_{L^p_{\text{avg}}(\Lambda)} := \left( \frac{1}{\# \Lambda} \sum_{x \in \Lambda} |f(x)|^p \right)^{1/p}. \]

ii) If \( \Omega \subseteq \mathbb{R}^n \) is measurable with finite Lebesgue measure and \( f \in L^p(\Omega) \), then
\[ \| f \|_{L^p_{\text{avg}}(\Omega)} := \left( \frac{1}{|\Omega|} \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}. \]
The definition of these averages extends in the obvious manner to the \( p = \infty \) case.
Lemma 8. To prove the local inequality (15) for some fixed $R \geq 1$ it suffices to show
\[
|\hat{G}|_{L^{p'}(B_R)} \leq R^{n-1} \|G\|_{L^p(S^{n-1}(R^{n-1}))} \tag{17}
\]
holds whenever $G$ is a smooth function supported in the annular region $S^{n-1}(R^{n-1})$.

Proof. Fix $R > 1$ and $\psi \in C^\infty_c(\mathbb{R}^n)$ with $\text{supp } \psi \subseteq B(0,1)$ and $|\psi(x)| \geq 1$ for all $x \in B(x_0, 1)$. Defining $G := \psi_{R^{-1}} \ast g\sigma$ where $\psi_{R^{-1}}(\xi) := R^n \psi(R \xi)$, it follows
\[
\|(g\sigma)^{\nu} \|_{L^{p'}(B(x_0, R))} \leq \|(g\sigma)^{\nu} \psi_{R^{-1}}\|_{L^{p'}(B(x_0, R))} = \|\hat{G}\|_{L^{p'}(B(x_0, R))}.
\]
Since $G$ is supported in $S^{n-1}(R^{n-1})$, we may apply (17) to deduce
\[
\|(g\sigma)^{\nu} \|_{L^{p'}(B(x_0, R))} \leq R^{n-1} \|\psi_{R^{-1}} \ast g\sigma\|_{L^p(S^{n-1}(R^{n-1}))}
\]
and it therefore remains to show
\[
\|\psi_{R^{-1}} \ast g\sigma\|_{L^{p'}(\mathbb{R}^n)} \leq R^{1/q'} \|g\|_{L^{p'}(\mathbb{R}^n)}.
\]
Observe if $q'$ replaced by 1, then the estimate
\[
\|\psi_{R^{-1}} \ast g\sigma\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)}
\]
is a simple consequence of Fubini’s theorem and so, by interpolation, the proof is concluded if we can show the corresponding estimate when $q'$ is replaced with $\infty$. That is, it suffices to show
\[
\|\psi_{R^{-1}} \ast g\sigma\|_{L^\infty(\mathbb{R}^n)} \leq R \|g\|_{L^\infty(\mathbb{R}^n)}.
\]
This in turn is reduced to showing
\[
\int_{\mathbb{R}^n} |\psi_{R^{-1}}(\xi - \eta)| d\sigma(\eta) \leq R \tag{18}
\]
uniformly in $\xi \in \mathbb{R}^n$. It is clear heuristically why (18) is true: the support of the integrand intersects $P^{n-1}$ on at most a $R^{-1} \times \cdots \times R^{-1}$ cap and the function has height at most $R^n$ leading to the base $\times$ height bound $R^{-(n-1)} \times R^n = R$. Turing to the rigorous proof of the estimate (18), we’ll in fact prove the more general statement that whenever $\psi$ is a Schwartz function on $\mathbb{R}^n$ and $S \subset \mathbb{R}^n$ is any compact hypersurface (no curvature conditions are required here), it follows
\[
\int_S |\psi_{R^{-1}}(\xi - \eta)| d\sigma(\eta) \leq R
\]
uniformly in $\xi \in \mathbb{R}^n$ and $R \gg 1$. By rapid decay the left-hand side of this expression can be bounded by
\[
I(\xi) := R^n \int_S \frac{d\sigma(\eta)}{(1 + R|\xi - \eta|)^n}.
\]
We form a dyadic decomposition of the above integral based on the size of the denominator appearing in the integrand. In particular, fix $\xi \in \mathbb{R}^n$ and consider partitioning $\mathbb{R}^n$ into the closed ball
\[
A_{-1}(\xi) := \{\eta \in \mathbb{R}^n : R|\xi - \eta| \leq 1\}
\]
and the dyadic annuli
\[
A_k(\xi) := \{\eta \in \mathbb{R}^n : 2^k < R|\xi - \eta| \leq 2^{k+1}\}
\]
defined for all $k \in \mathbb{N}$. Let $S_k(\xi) = A_k(\xi) \cap S$ for all $k \geq 0$ and write
\[
I(\xi) \leq R^n \sum_{k=-1}^{\infty} \left\| \int_{S_k(\xi)} \frac{d\sigma(\eta)}{(1 + R|\xi - \eta|)^n} \right\|.
\]
To estimate the terms of this sum we observe, due to the dimensionality of \( S \), that for \( r > 0 \) we have
\[
\sigma(B(\xi, r) \cap S) \lesssim r^{n-1}
\] (19)
Indeed, for \( r \) large the result is trivial whilst at small scales \((0 < r \ll 1)\) the surface is essentially flat and so, provided it is non-empty, \( B(\xi, r) \cap S \) is approximately a disc of radius \( r \), leading to the estimate in this case. The inequality (19) implies \( \sigma(S_{-1}(\xi)) \lesssim R^{-(n-1)} \) and \( \sigma(S_k(\xi)) \lesssim (R^{-1}2^k)^{n-1} \) for all \( k \geq 0 \) and so
\[
I(\xi) \lesssim R^n S_{-1}(\xi) + R^n \sum_{k=0}^{\infty} \frac{\sigma(S_k(\xi))}{2^kn} \lesssim R
\]
as required.

In fact, (15) and (17) are equivalent; the converse implication will not be used in the present notes but will be referred to later.

**Lemma 9.** If the local extension estimate (15) holds, then the inequality (17) is valid.

**Proof.** Without loss of generality (by the translation and rotation invariance of the problem, together with the triangle inequality) one may assume \( G \) is supported in \( \mathcal{N}_{R^{-1}}(P^{n-1}) \cap B(0,1/2) \). Consequently, \( \text{supp}G \) is contained in the disjoint union of vertical translates \( P_{\zeta}^{n-1} := P^{n-1} + (0,\zeta) \) of the paraboloid as \( \zeta \) varies over \((-R^{-1},R^{-1}) \subset \mathbb{R} \). By Fubini’s theorem and a simple change of variables,
\[
\tilde{G}(x) = \int_{|\zeta|<R^{-1}} \int_{|\xi|^2 \leq \zeta^2} G(\xi', |\xi'|^2 + \zeta) e^{2\pi i (x' \cdot \xi + x \cdot (|\xi'|^2 + \zeta))} d\xi' d\zeta
\]

\[
= \int_{|\zeta|<R^{-1}} (G\mid_{P_{\zeta}^{n-1}}d\sigma_{\zeta})^{\ast}(x) d\zeta,
\]
where \( \sigma_{\zeta} \) is the obvious measure on \( P_{\zeta}^{n-1} \).

Assuming the local extension estimate (15) holds, it immediately follows from translation invariance that
\[
\| (G\mid_{P_{\zeta}^{n-1}}d\sigma_{\zeta})^{\ast} \|_{L^{q'}(B_R)} \lesssim R^a |G\mid_{P_{\zeta}^{n-1}} \|_{L^p(P_{\zeta}^{n-1})}
\]
for all \( \zeta \). Combining the above estimate with Minkowski’s inequality one deduces that
\[
\| \tilde{G} \|_{L^{q'}(B_R)} \lesssim \int_{|\zeta|<R^{-1}} \| (G\mid_{P_{\zeta}^{n-1}}d\sigma_{\zeta})^{\ast} \|_{L^{q'}(B_R)} d\zeta \lesssim R^a \int_{|\zeta|<R^{-1}} \| G\mid_{P_{\zeta}^{n-1}} \|_{L^p(P_{\zeta}^{n-1})} d\zeta
\]
and Hölder’s inequality bounds the latter by
\[
R^a R^{-1/q'} \left( \int_{|\zeta|<R^{-1}} \| G\mid_{P_{\zeta}^{n-1}} \|_{L^{q'}(P_{\zeta}^{n-1})} d\zeta \right)^{1/q'} = R^{a-1} \| G\|_{L^{2q'/q}(\mathcal{N}_{R^{-1}}(P^{n-1}))^*}
\]
as required.

4. **Restriction estimates via Reverse Littlewood-Paley**

Here we discuss a possible approach to proving inequalities of the form (16) via a Littlewood-Paley theory for slabs on a neighbourhood of the paraboloid.

By our earlier discussion, to prove the restriction conjecture it suffices to show for any \( f \) with smooth Fourier transform support in \( \mathcal{N}_{R^{-1}}(P^{n-1}) \), the inequality
\[
\| f \|_{L^{2n/(n-1)}(B_R)} \lesssim R^{-1} \| \hat{f} \|_{L^2_{\text{avg}}(\mathcal{N}_{R^{-1}}(P^{n-1})))}
\]
holds. We’ll make the situation slightly simpler by fixing $f$ to be a smooth function whose Fourier transform belongs to the unit ball of $L^2(N_{R^{-1}}(P^{n-1}))$ and aiming to prove
\[ \|f\|_{L^{2n/(n-1)}(B_R)} \lesssim R^{-1}. \]
By Hölder’s inequality, this is clearly a weaker statement; it turns out, however, that symmetry considerations show the two estimates are in fact equivalent, but this is a technical point and we won’t discuss it here.

Arguing in a similar manner to our proof of Restriction $\implies$ Kakeya, we first decompose our neighbourhood of the parabola into a collection of essentially disjoint curved regions (which we will refer to as slabs) $\theta$ which are each contained within a $\sim R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1}$ rectangle. An explicit way to do this is to cover $[-1,1]^{n-1}$ with $2R^{-1/2} \times \cdots \times 2R^{-1/2}$ cubes $\{Q\}$ whose centres lie in the lattice $R^{-1/2} \mathbb{Z}^{n-1}$ and define each $\theta$ by
\[ \theta = \{(\xi', \eta + |\xi'|^2) : \xi' \in Q, |\eta| \lesssim R^{-1}\} \]
for some choice of $Q_{\theta} \in \{Q\}$. It is important to note in our construction the slabs do not have disjoint interiors: the overlap is included for technical reasons\footnote{In particular, it allows one to construct a partition of unity for $N_{R^{-1}}(P^{n-1})$ adapted to the family of slabs.} which will become manifest much later in the discussion. Furthermore, letting $\Omega$ denotes the set of normals to these slabs (that is, the set of vectors which correspond to the unit normal to the paraboloid at the centre of a slab), an important consequence of this construction is that these normals are $\sim R^{-1/2}$-separated.

Henceforth, we will let $f_\theta$ denote the Fourier restriction of $f$ to $\theta$; that is,
\[ \hat{f}_\theta := \hat{f} \chi_{\theta}. \]
With this new notation observe
\[ f \sim \sum_{\theta: \text{slab}} f_\theta, \]
where the summation is over the entire collection of slabs. Indeed, if the slabs had disjoint interiors then this identity would be valid with equality, but almost every point in $N_{R^{-1}}(P^{n-1})$ lies in some fixed number of slabs $C_n > 1$ which corresponds to the implied constant in the above equation. Thus, we wish to establish
\[ \left\| \sum_{\theta: \text{slab}} f_\theta \right\|_{L^{2n/(n-1)}(B_R)} \lesssim R^{-1}. \]
In fact, we’ll aim to prove the (at least ostensibly) stronger inequality
\[ \left\| \sum_{\theta: \text{slab}} f_\theta \right\|_{L^{2n/(n-1)}(\mathbb{R}^n)} \lesssim R^{-1}; \quad (20) \]
indeed, fattening the paraboloid on the frequency side encapsulates the spatial localisation and so this global estimate is essentially no more difficult to prove than the local version.

\footnote{Indeed, if $(\xi_j', |\xi_j'|^2) \in P^{n-1}$ for $j = 1, 2$ and $\nu_j$ denotes the unit normal at $(\xi_j', |\xi_j'|^2)$, then
\[ \nu_1 \cdot \nu_2 = \frac{4\xi_1' \cdot \xi_2' + 1}{(4|\xi_1'|^2 + 1)^{1/2}(4|\xi_2'|^2 + 1)^{1/2}}. \]
Suppose $|\xi_1' - \xi_2'| \gtrsim R^{-1/2}$ and, without loss of generality, assume the above dot product is positive (otherwise $\nu_1$ and $\nu_2$ are $O(1)$-separated). Then
\[ \nu_1 \cdot \nu_2 \leq \left( 1 - \frac{4(|\xi_1'|^2 + 1)(4|\xi_2'|^2 + 1)}{4(|\xi_1'|^4 + 1)(4|\xi_2'|^4 + 1)} \right)^{1/2} \leq (1 - CR^{-1})^{1/2} \]
and it follows that $|\nu_1 - \nu_2|^2 = 2(1 - \nu_1 \cdot \nu_2) \gtrsim R^{-1}$, by the mean value theorem.}
We would like to estimate the left-hand sum in (20) in terms of individual contributions from the $f_\ell$; we’ll see later that these individual portions basically behave like a superposition of many ‘parallel’ Knapp examples and can be relatively easily understood.\footnote{That is, provided we have at our disposal pretty heavy-weight tools such as the full Kakeya conjecture! Here we really interested in understanding the cancellation in the problem and are therefore rather cavalier when it comes to “pure size estimates”.} The main difficulty here is to understand the cancellation between the $f_\ell$. One approach would be to try to replace $|\sum_\ell f_\ell|$ with an $\ell^2$ expression, in the manner of (11) from the Restriction $\implies$ Kakeya argument. This has the effect of separating the contributions from the individual $f_\ell$ whilst accounting for any destructive interference between terms. In particular, we would like an inequality of the following form:

**Conjecture 10** (Reverse Littlewood-Paley inequality for slabs). Suppose $f$ has frequency support in $\mathcal{N}_{p^{-1}}(P^{n-1})$. With the above notation,

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \left( \sum_{\theta: R^{-1/2} - \text{slab}} |f_\theta|^2 \right)^{1/2} \|f\|_{L^p(\mathbb{R}^n)}$$

(21) holds whenever $2 \leq p \leq 2n/(n-1)$.

Thus, the Littlewood-Paley (or square-function) estimate (21) would tell us there is significant destructive interference present between the different frequency localised parts of $f$: enough to allow one to improve the trivial $\ell^1$ inequality $\|f\|_p \leq \|\sum_\ell f_\ell\|_p$, to an $\ell^2$ bound.

It is easy to see the $p = 2$ case of the conjecture follows immediately from the almost-orthogonality of the $f_\ell$.

**Lemma 11.** If $f$ has frequency support in $\mathcal{N}_{p^{-1}}(P^{n-1})$, then

$$\|f\|_{L^2(\mathbb{R}^n)} \lesssim \left( \sum_{\theta: R^{-1/2} - \text{slab}} |f_\theta|^2 \right)^{1/2} \|f\|_{L^2(\mathbb{R}^n)}.$$ 

**Proof.** By Plancherel’s theorem,

$$\|f\|_{L^2(\mathbb{R}^n)} \sim \left\| \sum_{\theta: R^{-1/2} - \text{slab}} f_\theta \right\|_{L^2(\mathbb{R}^n)} = \left\| \sum_{\theta: R^{-1/2} - \text{slab}} \hat{f}_\theta \right\|_{L^2(\mathbb{R}^n)}$$

and Cauchy-Schwarz bounds this by

$$\left( \sum_{\theta: R^{-1/2} - \text{slab}} |f_\theta|^2 \right)^{1/2} \left( \sum_{\theta: R^{-1/2} - \text{slab}} \chi_\theta \right)^{1/2} \left\| \sum_{\theta: R^{-1/2} - \text{slab}} \chi_\theta \right\|_{L^2(\mathbb{R}^n)}.$$ 

Since the slabs are finitely-overlapping the latter expression is dominated by (a constant multiple of)

$$\left( \sum_{\theta: R^{-1/2} - \text{slab}} |f_\theta|^2 \right)^{1/2} \left\| \sum_{\theta: R^{-1/2} - \text{slab}} \chi_\theta \right\|_{L^2(\mathbb{R}^n)} = \left( \sum_{\theta: R^{-1/2} - \text{slab}} |f_\theta|^2 \right)^{1/2} \|f\|_{L^2(\mathbb{R}^n)},$$

where we have once again applied Plancherel’s theorem (after reordering the norms). □

The $\ell^2$-decoupling inequalities studied in these lectures are closely related to and partially motivated by (21). At present, however, we will simply assume Conjecture 10 holds and examine the consequences for the Fourier restriction problem.

We have now isolated the function $f$ into frequency localised parts $f_\ell$ and have controlled their oscillatory interactions via the above Littlewood-Paley estimate. This more-or-less takes care of the significant cancellation in the problem and what remains are pure size considerations. Returning to the uncertainty principal, since the function $f_\ell$ is frequency supported in essentially a $\sim R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1}$
rectangle, it should be essentially constant on rectangles with reciprocal dimensions. In order to make this statement rigorous, we employ what is known as a wave packet decomposition of $f_\theta$.

To introduce the wave packet decomposition, we first need to define wave packets. These objects are essentially smoothed out copies of the Knapp example introduced earlier. Fix a bump function $\phi$ whose Fourier transform is supported on $[-1/2, 1/2]^n$ and equals 1 on $[-1/4, 1/4]^n$. For any rectangle $T$ let $a_T$ denote an affine transformation whose linear part has determinant $\sim |T|$ which maps $[-1/4, 1/4]^n$ to $T$ bijectively and define $\phi_T := \phi \circ a_T^{-1}$.

We will consider rectangles orientated in directions determined by the slabs. In particular, suppose $\omega$ is the normal direction to $P^{n-1}$ at the centre $\xi_\theta$ of the slab $\theta$ (in this situation we say $\theta$ has normal $\omega$). We let $\Theta(\theta)$ denote a finitely-overlapping collection of $\sim R^{1/2} \times \cdots \times R^{1/2} \times R$ rectangles which cover $\mathbb{R}^n$ and are orientated in the direction of $\omega$.

**Definition** (Wave packet adapted to $T$). The wave packet adapted to $T \in \Theta(\theta)$ is the function given by

$$\psi_T(x) := |T|^{-1}e^{2\pi i \xi_\theta \cdot x} \phi_T(x).$$

Notice that (provided the implicit constants are suitably chosen) $\hat{\psi}_T$ is supported in a dilate of $\theta$ and has modulus 1 on $\theta$. Indeed, a simple computation yields

$$|\hat{\psi}(\xi)| \sim |\hat{\phi}(a_T^{-1}(\xi - \xi_\theta))|$$

where $a_T^*$ is the adjoint of the linear part of $a_T$ and, consequently,

$$\{ \xi \in \hat{\mathbb{R}}^n : |\hat{\psi}(\xi)| = 1 \} \subseteq (a_T^*)^{-1}([-1/4, 1/4]^n) + \xi_\theta.$$

It is easy to see $(a_T^*)^{-1}([-1/4, 1/4]^n)$ is a rectangle centred at the origin which is dual to $T$. Furthermore, if the sidelengths of $T$ are chosen correctly (depending only on some innocuous constant), then $\theta$ is contained in the translate of this dual rectangle by $\xi_\theta$, as required. A similar argument can be used to show the support condition holds.

**Lemma 12** (Wave packet decomposition). Let $f$ be a smooth function on $\mathbb{R}^n$. For any slab $\theta$ there exists a decomposition

$$f_\theta(x) = \sum_{T \in \Theta(\theta)} f_T \psi_T(x)$$

where the constants $f_T$ satisfy

$$\left( \sum_{T \in \Theta(\theta)} |f_T|^2 \right)^{1/2} \leq \| \hat{f}_\theta \|_{L^2_{\lambda_{\mathbb{R}^n}}(\theta)}.$$

**Proof.** Let $T_0$ denote the $\sim R^{1/2} \times \cdots \times R^{1/2} \times R$ rectangle which is orientated in the direction of the normal to the slab $\theta$ and centred at 0. It follows that, provided the implied constant defining the dimensions of $T_0$ is chosen correctly,

$$g_\theta(\xi) := \hat{f}_\theta((a_T^*)^{-1}(\xi + \xi_\theta))$$

is supported in $[-1/4, 1/4]^n$ and so we can think of it as a function on the torus $\mathbb{T}^n \cong [-1/2, 1/2]^n$. Expressing this function in terms of its Fourier series we deduce the formula

$$\hat{f}_\theta(\xi) = \sum_{k \in \mathbb{Z}^n} u_k e^{-2\pi i k \cdot (\xi + \xi_\theta)} \quad \text{for } \xi \in (a_T^*)^{-1}([-1/2, 1/2]^n) + \xi_\theta.$$

Here the $\{u_k\}_{k \in \mathbb{Z}}$ are the Fourier coefficients of $g_\theta$ which satisfy

$$\left( \sum_{k \in \mathbb{Z}} |u_k|^2 \right)^{1/2} = \| g_\theta \|_{L^2([-1/2, 1/2]^n)} \leq \| \hat{f}_\theta \|_{L^2_{\lambda_{\mathbb{R}^n}}(\theta)}.$$
By our earlier observations, the function \( \xi \mapsto \hat{\phi}(a_{T_0}^*(\xi - \xi_\theta)) \) equals 1 on the support of \( \hat{f}_0 \) and is itself supported in \( (a_{T_0}^*)^{-1}([-1/2,1/2]^n) + \xi_\theta \) so that
\[
\hat{f}_0(\xi) = \sum_{k \in \mathbb{Z}^n} u_k e^{-2\pi i k \cdot a_{T_0}^*(\xi - \xi_\theta)} \hat{\phi}(a_{T_0}^*(\xi - \xi_\theta)),
\]
which is valid on the whole of \( \hat{\mathbb{R}}^n \). Taking the inverse Fourier transform,
\[
f_0(x) = \sum_{k \in \mathbb{Z}^n} u_k |\det a_{T_0}^{-1}| e^{2\pi i x \cdot \xi_\theta} \phi_{T_0}(x - a_{T_0}k) = c \sum_{k \in \mathbb{Z}^n} u_k \psi_{T_0 + a_{T_0}k}(x),
\]
and the proof is concluded by defining the \( T(\theta) \) to be the collection of rectangles of the form \( T_0 + a_{T_0}k \).

Applying the conjectured Littlewood-Paley estimate together with the wave-packet decomposition, we now wish to bound
\[
R^{-(n+1)/2} \left( \sum_{T \in \mathcal{T}(\theta)} \sum_{l \in \mathbb{Z}^n} |\phi_T(l)| |\phi_T(l)| \right)^{1/2} \left\| \left( \mathbb{E}_{T \in \mathcal{T}(\theta)} |f_T| |\chi_T| \right)^{1/2} \right\|_{L^{2n/(n-1)}(\mathbb{R}^n)},
\]
(22)

The Schwartz function \( \phi_T \) rapidly decays away from \( T \); we would like to replace this “smooth indicator function” with the sharp cut-off \( \chi_T \) in order to place ourselves in position to apply the (hypothesised) Kakeya estimates. Of course, \( \phi_T \) is not compactly supported and a modicum of extra work is required to make such a “substitution” rigorous. Let \( \chi_{T,l} \) denote the characteristic function of the rectangle \( A_T([-1/4,1/4]^n + l/2) \) for each \( l \in \mathbb{Z}^n \) so that the \( \{\chi_{T,l}\}_{l \in \mathbb{Z}^n} \) form a rough partition of unity of \( \mathbb{R}^n \). By the rapid decay of \( \phi \) we have
\[
|\phi_T(l)| = \sum_{l \in \mathbb{Z}^n} |\phi_T(l)| |\chi_{T,l}(x) | \leq \sum_{l \in \mathbb{Z}^n} \frac{\chi_{T,l}(x)}{(1 + |a_{T_0}^{-1}(x)|)^{n+1}} \leq \sum_{l \in \mathbb{Z}^n} \frac{\chi_{T,l}(x)}{(1 + |l|)^{n+1}}.
\]
We combine the preceding observations together with a two-fold application of Minkowski’s inequality to deduce (22) is bounded above by (a constant multiple of)
\[
R^{-(n+1)/2} \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-(n+1)} \left( \sum_{T \in \mathcal{T}(\theta)} \sum_{l \in \mathbb{Z}^n} |f_T| |\chi_{T,l}| \right)^{1/2} \left\| \left( \mathbb{E}_{T \in \mathcal{T}(\theta)} |f_T| |\chi_T| \right)^{1/2} \right\|_{L^{2n/(n-1)}(\mathbb{R}^n)}.
\]
Since the supports of the \( \chi_{T,l} \) are essentially disjoint as \( T \) varies over \( \mathcal{T}(\theta) \), for a fixed value of \( l \) the \( L^{2n/(n-1)} \)-norm in the above expression can be bounded by
\[
\left\| \sum_{T \in \mathcal{T}(\theta)} \sum_{l \in \mathbb{Z}^n} |f_T| |\chi_{T,l}| \right\|_{L^{2n/(n-1)}(\mathbb{R}^n)}^{1/2}.
\]
(23)
To conclude the proof, it suffices to show that (23) is \( O(R^{(n-1)/2}) \); one may then sum in \( l \) to obtain the desired restriction inequality.

Recall, the properties of the wave packet decomposition and our initial hypothesis on \( f \) imply
\[
\sum_{T \in \mathcal{T}(\theta)} |f_T|^2 \leq 1
\]
for each \( \theta \). One may therefore find sequences \( (c_T)_{T \in \mathcal{T}(\theta)} \) of non-negative real numbers such that
\[
\sum_{T \in \mathcal{T}(\theta)} c_T = 1
\]
and such that (23) is dominated by
\[
\left\| \sum_{\theta : R^{-1/2} - \text{slab}} \sum_{T \in \mathcal{T}(\theta)} c_T \chi_{T,l} \right\|_{L^{n/(n-1)}(\mathbb{R}^n)}^{1/2},
\] (24)

Consider randomly selecting a sequence of rectangles, one for each direction $\theta$ where each $T$ is chosen from $\mathcal{T}(\theta)$ with probability $c_T$. This corresponds to endowing the space $\prod_\theta \mathcal{T}(\theta)$ with the probability measure that assigns the probability $\prod_\theta c_T$ to each singleton \{$(T_\theta)$\}. For fixed $x \in \mathbb{R}^n$ consider the random variable $\sum_{\theta} \chi_{T_\theta,l}(x)$ which counts the number of rectangles in a randomly selected sequence $(T_\theta)$ for which $x \in \text{supp} \chi_{T_\theta,l}$. It is easy to see, using the linearity of the expectation, that
\[
E \left[ \sum_{\theta : R^{-1/2} - \text{slab}} \chi_{T_\theta,l}(x) \right] = \sum_{\theta : R^{-1/2} - \text{slab}} \sum_{T \in \mathcal{T}(\theta)} \chi_{T_\theta,l}(x)
\]
and so, by Minkowski’s inequality,
\[
\left\| \sum_{\theta : R^{-1/2} - \text{slab}} \sum_{T \in \mathcal{T}(\theta)} c_T \chi_{T,l} \right\|_{L^{n/(n-1)}(\mathbb{R}^n)} \lesssim E \left[ \sum_{\theta : R^{-1/2} - \text{slab}} \chi_{T_\theta,l} \right]^{1/2}.
\]
The argument is concluded by applying the hypothesised Kakeya estimate
\[
\left\| \sum_{\theta : R^{-1/2} - \text{slab}} \chi_{T_\theta,l} \right\|_{L^{n/(n-1)}(\mathbb{R}^n)} \lesssim R^{n-1},
\]
which is valid for every choice of $l \in \mathbb{Z}^n$ and $(T_\theta) \in \prod_\theta \mathcal{T}(\theta)$.

5. Progress on the Littlewood-Paley inequality for slabs

We have already seen that a global version of the Littlewood-Paley inequality for slabs is true (and somewhat trivial) for $p = 2$ in all dimensions. In this section a classical argument is presented which proves (21) for $n = 2$ and $p = 4$. This can be combined with the above analysis to give a full proof of the restriction conjecture for $n = 2$. In later notes we will present a simpler proof of two-dimensional restriction relying on bilinear techniques.

Proposition 13 (Córdoba and Fefferman). Conjecture 10 holds when $n = 2$. In particular, with the above notation,
\[
\|f\|_{L^4(\mathbb{R}^2)} \lesssim \left( \sum_{\theta : R^{-1/2} - \text{slab}} |f_\theta|^2 \right)^{1/2},
\]
whenever $f$ has Fourier support in $N_{R^{-1}}(P^1)$. (So in this case the result holds without any $\varepsilon$-leakage in the $R$-exponent).

Proof. The Fourier support condition ensures
\[
\|f\|_{L^4(\mathbb{R}^2)} \sim \left\| \sum_{\theta, \theta' : R^{-1/2} - \text{slab}} f_\theta \bar{f}_{\theta'} \right\|_{L^2(\mathbb{R}^2)}^2
\]
and, due to the trivial Cauchy-Schwarz estimate
\[
\left\| \sum_{\theta, \theta' : R^{-1/2} - \text{slab}} f_\theta \bar{f}_{\theta'} \right\| \lesssim \left\| \sum_{\theta, \theta' : R^{-1/2} - \text{slab}} |f_\theta|^2 \right\|^{1/2},
\]
it is enough to bound the right-hand side of (25) with the summation restricted to pairs of well-separated slabs $(\theta, \theta')$ for which $\text{dist}(\theta, \theta') \gtrsim R^{-1/2}$. In particular, it suffices to show
\[
\left\| \sum_{\theta, \theta' : R^{-1/2} - \text{slab}} f_\theta \bar{f}_{\theta'} \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| \sum_{\theta, \theta' : R^{-1/2} - \text{slab}} |f_\theta|^2 \right\|_{L^2(\mathbb{R}^2)}.
\]
which can be interpreted as the statement that the $f_0 f_{\theta'}$ are almost orthogonal.

Observe
\[
\text{supp } f_0 \ast f_{\theta'} \subseteq \theta - \theta'
\]
and, by Plancherel, the problem now reduces to showing these Minkowski differences have bounded overlap as $(\theta, \theta')$ varies over well-separated pairs.

Suppose $\theta_1 - \theta'_1 \cap \theta_2 - \theta'_2 \neq \emptyset$ for $R^{-1/2}$-slabs $\theta_1, \theta'_1$ which satisfy $\text{dist}(\theta_1, \theta'_1) \gtrsim R^{-1/2}$ for $j = 1, 2$. Thus, there exist $y_j \in \theta_j$ and $y'_j \in \theta'_j$ for $j = 1, 2$ such that
\[
y_1 - y'_1 - y_2 + y'_2 = 0
\]
Since each slab belongs to $\mathcal{N}_{R^{-1}}(P^1)$, there exist $t_j, t'_j \in [-1, 1]$ such that
\[
|y_j - (t_j, t'_j)|, |y'_j - (t'_j, (t'_j)^2)| < R^{-1}
\]
for $j = 1, 2$ and so
\[
|t_1 - t'_1| - (t_2 - t'_2)| \lesssim R^{-1};
\]
\[
|t_2^2 - (t'_2)^2| - (t_2^2 - (t'_2)^2)| \lesssim R^{-1}.
\]

It follows that
\[
|t_1 - t'_1| - (t_2 + t'_2)| \lesssim R^{-1}
\]
and the separation condition on the $\theta_j, \theta'_j$ then implies
\[
|t_1 + t'_1| - (t_2 + t'_2)| \lesssim R^{-1/2}.
\]
Comparing (26) and (27), one deduces $|t_1 - t_2|, |t'_1 - t'_2| \lesssim R^{-1/2}$ and therefore
\[
|y_1 - y_2|, |y'_1 - y'_2| \lesssim R^{-1/2}.
\]
Consequently, given $\theta_1, \theta'_1$ there are at most $O(1)$ choices of $\theta_2, \theta'_2$ for which $\theta_1 - \theta'_1 \cap \theta_2 - \theta'_2 \neq \emptyset$ and
\[
\#\{ (\theta, \theta') : \text{dist}(\theta, \theta') \gtrsim R^{-1/2} \text{ and } \xi \in \theta - \theta' \} \lesssim 1
\]
for all $\xi \in \mathbb{R}^2$, as required. \hfill \Box

In higher dimensions partial results are known, which are typically non-optimal in the $R$ exponent. We mention, for instance, the work of Bourgain [2] which shows (21) holds for the wider range of $\theta, \theta'$.

6. The $\ell^2$ decoupling conjecture

We saw in the previous sections that, if true, a Littlewood-Paley inequality for slabs, (21), would provide an effective tool for understanding significant cancellation phenomena in the Fourier restriction problem. We remark that an argument of Carbery [5] shows that (21) would also imply the Kakeya conjecture and, consequently, the preceding argument “reduces” proving the restriction problem to establishing Conjecture 10.\(^{13}\)

For the remainder of these lectures we will investigate a weaker variant of (21); namely, we are interested in estimates of the form
\[
\|f\|_{L^p(R^n)} \lesssim \left( \sum_{\theta : R^{-1/2} \text{-slab}} \|f_\theta\|_{L^p(R^n)}^2 \right)^{1/2}.
\]

\(^{12}\)The exponent $2(n + 1)/(n - 1)$ pervades restriction theory and, in particular, forms the endpoint for the Stein-Tomas theorem discussed earlier.

\(^{13}\)Attempting to prove the whole restriction conjecture from this direction may be a somewhat optimistic strategy: Conjecture 10 appears to be very powerful and in all likelihood considerably more difficult than the restriction conjecture.
Using the terminology introduced by Bourgain and Demeter, we will refer to (28) as an $\ell^2$-decoupling inequality. The idea is that, as with the square function estimate, this inequality separates (or “decouples”) the contributions to $\|f\|_p$ from the many frequency localised portions $f_\theta$. This is done in an efficient way, taking into account the cancellation between the $f_\theta$. In this regard, however, the decoupling inequalities are, a priori, not quite as effective as the Littlewood-Paley inequality and, indeed, (28) is clearly weaker than (21) for $2 \leq p \leq 2n/(n - 1)$ by Minkowski’s inequality.

We remark that (28) does not act as a substitute for the square function estimate in the sense that it is not clear if it is possible to modify our earlier arguments to prove the restriction conjecture using decoupling inequalities. Decoupling theory does, however, have a plethora of applications and [4] uses (28) to not only study restriction theory, but problems in PDE, additive combinatoric and number theory.

Given the square-function conjecture, it makes sense to conjecture the following:

**Conjecture 14 ($\ell^2$-decoupling conjecture, preliminary version).** With the above notation,

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \left( \sum_{\theta : R^{-1/2} - \text{slab}} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}.$$  

holds whenever $2 \leq p \leq 2n/(n - 1)$.

In the next few lectures we’ll present a proof of this conjecture. The key tool will be multilinear restriction theory, which is well understood thanks to the work of Bennett, Carbery, Tao and Guth, et al. We mention that the above conjecture was known prior to Bourgain and Demeter’s paper, appearing in [3].

It turns out that the range of exponents $2 \leq p \leq 2n/(n - 1)$ is no longer optimal when considering the weaker decoupling inequalities and a more appropriate endpoint is the Stein-Tomas exponent $2(n + 1)/(n - 1)$. For larger values of $p$ one may still obtain decoupling inequalities, but necessarily with a constant depending polynomially on $R$. The full version of the conjecture is the following.

**Conjecture 15 ($\ell^2$-decoupling conjecture, full version).** With the above notation,

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim R^{\alpha(p)} \left( \sum_{\theta : R^{-1/2} - \text{slab}} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}$$

holds with

$$\alpha(p) := \begin{cases} 0 & \text{if } 2 \leq p \leq 2(n + 1)/(n - 1); \\ (n - 1)/4 - (n + 1)/2p & \text{if } 2(n + 1)/(n - 1) < p. \end{cases}$$

The above corollary can be seen to be best possible, except possibly for the $\epsilon$-loss in $R$.

We leave it as an exercise to show no decoupling estimates are possible for $p < 2$.

Recall that prior to Bourgain and Demeter’s work it was shown in [3] that the conjecture holds for $2 \leq p \leq 2n/(n - 1)$. In addition, partial results in the “super-critical” regime $p \geq 2(n + 1)/(n - 1)$ were established in [7] and [6]. Bourgain and Demeter [4] were able to adapt and develop multilinear restriction arguments to establish the full super-critical range of estimates from which one may establish Conjecture 15 in full.\[14\] In the applications it is often necessary to have the full power of Conjecture 15 and, after discussing the preliminary Conjecture 14, we will detail how the complete range of estimates was proved in later lectures.

\[14\]The sub-critical estimates follow from the $p = 2(n + 1)/(n - 1)$ case together with the trivial $p = 2$ inequality - the details of this argument will be discussed in later lectures.
REFERENCES

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