In the following take $1 < p, q < \infty$ otherwise specified, and $(X, |\cdot|)$ a \( \sigma \)-finite measure space with no atoms.

The usual definition of Lorentz space is as follows:

**Definition 1.** The space \( L^{p,q}(X) \) is the space of measurable functions \( f \) such that
\[
\|f\|_{L^{p,q}(X)} := \left( \int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t} \right)^{1/q} < \infty,
\]
where \( f^* \) is the decreasing rearrangement\(^1\) of \( f \). If \( q = \infty \) then define instead
\[
\|f\|_{L^{p,\infty}(X)} := \sup_t t^{1/p} f^*(t) < \infty.
\]

A few things are to be noticed here. First of all, \( \|\cdot\|_{L^{p,q}(X)} \) is a quasi-norm. This follows from known properties of rearrangements: indeed, \((f + g)^*(2t) \leq f^*(t) + g^*(t)\), thus
\[
\|f + g\|_{L^{p,q}(X)}^q = \int_0^\infty (2t)^{q/p} (f + g)^*(2t)^q \frac{dt}{t}
\]
\[
\lesssim_{p,q} \int_0^\infty t^{q/p} (f^*(t)^q + g^*(t)^q) \frac{dt}{t} = \|f\|_{L^{p,q}}^q + \|g\|_{L^{p,q}}^q,
\]
and the constants implied are bigger than 1. Another thing to notice is that \( L^{p,p}(X) = L^p(X) \), as
\[
\|f\|_{L^{p,p}(X)}^p = \int_0^\infty f^*(t)^p \frac{dt}{t} = \int_0^\infty t^{p-1} |\{f^* > t\}| dt
\]
\[^1\text{that is, } f^*(t) := \inf\{s : |\{f > s\}| \leq t\}. \text{ This implies immediately that } |\{f > f^*(t)\}| \leq t. \text{ Another useful equality is } |\{f > t\}| = |\{s : f^*(s) > t\}|.\]
\[
\int_0^\infty t^{p-1} \{|f| > t\} \, dt = \int |f|^p \, dx = \|f\|_{L^p(X)}^p.
\]

The spaces \(L^{p,\infty}\) are the usual weak \(L^p\) spaces: for small \(\varepsilon > 0\) by definition of \(f^*\) it follows indeed that \(f^* (|\{|f| > \lambda\}| - \varepsilon) > \lambda\), from which

\[
\sup_t t^{1/p} f^*(t) \geq (|\{|f| > \lambda\}| - \varepsilon)^{1/p} f^* (|\{|f| > \lambda\}| - \varepsilon) > \lambda (|\{|f| > \lambda\}| - \varepsilon)^{1/p}
\]

and by taking the limit \(\varepsilon \to 0\) one recovers the usual weak \(L^p\) norm; as for the opposite direction, it is \(|\{|f| > f^*(t) - \varepsilon\}| > t\), and thus

\[
t^{1/p} (f^*(t) - \varepsilon) < |\{|f| > f^*(t) - \varepsilon\}|^{1/p} (f^*(t) - \varepsilon) \leq \sup \lambda |\{|f| > \lambda\}|^{1/p} \lambda,
\]

and by taking the limit \(\varepsilon \to 0\) again one concludes the two definitions agree.

An important inequality is the so called Hardy-Littlewood inequality for rearrangements,

**Theorem 1** (Hardy-Littlewood inequality). For any \(f, g\) measurable and vanishing at infinity one has

\[
\int |f(x)g(x)| \, dx \leq \int_0^\infty f^*(t) g^*(t) \, dt.
\]

**Proof.** By the layer cake representation (see below) one can write

\[
|f(x)| = \int_0^\infty \chi_{\{|f(x)| \geq s\}} \, ds,
\]

and therefore

\[
\int_X |f(x)g(x)| \, dx = \int_X \int_0^\infty \int_0^\infty \chi_{\{|f(x)| \geq s\}} \chi_{\{|g(x)| \geq t\}} \, dt \, ds \, dx
\]

\[
= \int_X \int_0^\infty \int_0^\infty \chi_{\{|f(x)| \geq s\} \cap \{|g(x)| \geq t\}} \, dt \, ds \, dx,
\]

which by Fubini is

\[
= \int_0^\infty \int_0^\infty \left|\{|f(x)| \geq s\} \cap \{|g(x)| \geq t\}\right| \, dt \, ds
\]

\[
\leq \int_0^\infty \int_0^\infty \min\{|\{|f(x)| \geq s\}|, \{|g(x)| \geq t\}| \, dt \, ds
\]
but now the sets are one contained inside the other thus we can substitute the minimum by the intersection,

\[
\int_0^\infty \int_0^\infty |\{f^* \geq s\} \cap \{g^* \geq t\}| \, dt \, ds
\]

and by Fubini again, since \(\int_0^\infty \chi_{\{f^* \geq s\}}(u) \, ds = f^*(u)\),

\[
= \int_0^\infty f^*(u)g^*(u) \, du.
\]

From this last inequality it follows that the dual of \(L^{p,q}\) is \(L^{p',q'}\):

**Proposition 1** (Duality of Lorentz spaces). It is

\[(L^{p,q})^* = L^{p',q'}\]

*Proof.* In one direction, notice that by Hölder it is simply

\[
\left| \int_X f g \, dx \right| \leq \int_0^\infty f^*g^* \, dt \leq \left( \int_0^\infty t^{q/p} f^*(t)^q \, dt \right)^{1/q} \left( \int_0^\infty t^{q'/p'} g^*(t)^{q'} \, dt \right)^{1/q'}.
\]

The opposite direction is a bit more complicated but we reproduce it here briefly for the sake of completeness. The interested reader can find more details in Grafakos’ Classical Fourier Analysis [2] (although a big part of it is given as an exercise, with a reasonable amount of hints). A slightly different proof is in [2]. We need a few facts: first of all that for \(\alpha > 0\) and \(p \geq 1\)

\[
\int_0^\infty x^{\alpha} \left( \int_X |f(y)| \, dy \right)^p \, dx \lesssim_p \int_0^\infty |f(x)|^p x^{p+\alpha} \frac{dx}{x};
\]

(1) to prove it you can consider the multiplicative group \((\mathbb{R}^+, \cdot)\) and its Haar measure \(\frac{dy}{y}\), take the convolution of \(|f(x)| x^{1+b/p}\) with \(x^{b/p} \chi_{[0,1]}(x)\) (w.r.t. to the group!), and finish by Young’s inequality.

3
Next we need to know a refinement of Hardy-Littlewood inequality: namely that
\[
\sup_{h \sim f} \int |hg| \, dx = \int_0^\infty f^*(t)g^*(t) \, dt,
\]
where \( h \sim f \) means that \( h \) and \( f \) are equidistributed: \( \{|h| > \lambda\} = \{|f| > \lambda\} \) for any \( \lambda \). This can be proven by observing that, since the measure is non-atomic, for any set \( A \) of finite non-zero measure we can find a set \( \tilde{A} \) s.t. \( |A| = |\tilde{A}| \) and \( \int_{\tilde{A}} |g| \, dx = \int_0^{|A|} g^*(t) \, dt \). To prove it, fix \( |A| = s \) and take \( \{|g| \geq g^*(s)\} \supset A \supset \{|g| > g^*(s)\} \) with measure \( |A| \) - which you can find, because there are no atoms. Then layer cake representation applied to \( g\chi_{\tilde{A}} \) proves the claim. With this one can prove (2) by applying it to the case of \( f \) positive simple function. Then \( f = \sum_j a_j \chi_{E_j} \) with \( a_j \) all positive, and if \( b_j = a_j - a_{j+1} \) one can write \( f = \sum_j b_j \chi_{F_j} \) with \( F_j \) nested, namely \( F_j := E_1 \cup \ldots \cup E_j; \) and by taking the sets \( \tilde{F}_j \) as before (i.e. \( |\tilde{F}_j| = |F_j| \) and \( \int_{\tilde{F}_j} |g| = \int_{|F_j|} g^* \) we can form \( \tilde{f} = \sum_j b_j \chi_{\tilde{F}_j} \) for which \( (\tilde{f})^* = f^* \), and verify that
\[
\int_X |\tilde{f}g| \, dx = \int_0^\infty f^*(t)g^*(t) \, dt.
\]
For a general \( f \) use approximation, since the simple functions are dense in \( L^{p,q} \) for \( q < \infty \).

Now we’re ready to finish the proof. By usual considerations, akin to those needed to conclude duality for \( L^p \) spaces, we can see that by Radon-Nykodim any functional in \( (L^{p,q})^* \) is given by integration against a measurable function. Fix such a linear functional and call \( g \) the associated function and \( C_g \) the operator norm of \( g \) as a linear functional. We need to prove \( g \in L^{p',q'} \). To do so, take a function \( f \) whose rearrangement is \( f^*(t) = \int_{t/2}^\infty s^{q'/q'-1} g^*(s) \, ds \), which you can do (again, prove it on simple functions for a generic choice to convince yourself; notice \( f^* \) is decreasing by definition). Then, by formula [1]
\[
\|f\|_{L^{p,q}}^q = \int_0^\infty t^{q/p} \left( \int_{t/2}^\infty s^{q'/q'-1} g^*(s) \, ds \right)^q \frac{dt}{t} \leq_p \int_0^\infty t^{q'/q'} g^*(t) \frac{dt}{t} = \|g\|_{L^{p',q'}}^{q'/q}
\]
(if finite). Now notice that by formula (2), since \( h \sim f \), it is
\[
\int f^*g^* \, dt \leq C_g \|f\|_{L^{p,q}} \leq_p C_g \|g\|_{L^{p',q'}}^{q'/q}.
\]
On the other hand

\[ \int f^* g^* \, dt \geq \int_0^\infty \int_{t/2}^t s^{q'/p'-1} g^*(s)^{q'-1} \, ds \, g^*(t) \, dt \]

\[ \gtrsim \int_0^\infty t^{q'/p'-1} g^*(t)^{q'-1} \, dt = \|g\|_{L^{p',q'}}^{q'}. \]

It follows that \( \|g\|_{L^{p',q'}} \lesssim C_g \), on the condition that the quasi-norm be finite. But the measure is \( \sigma \)-finite, and thus we can find a sequence of nested sets \( \Omega_n \) of finite measure covering all of \( X \), and on each one the quasi-norm is indeed finite and we get the result by taking the limit in \( n \).

We can state with no ambiguity that a linear (or sublinear) operator \( T \) is of strong type \((p, q)\) if it maps \( L^p \) to \( L^q \) boundedly, that it is of weak type \((p, q)\) if it maps \( L^p \) to \( L^{q,\infty} \) boundedly, and it is of restricted weak type if it maps \( L^{p,1} \) to \( L^{q,\infty} \) boundedly. The first two definitions agree with the classical ones by the discussion above. To see the last definition agrees with the usual one, one has to notice that the unit ball in \( L^{p,1} \) is the convex hull of normalized characteristic functions \( \frac{1}{|E|^{1/p}} \chi_E \). Indeed \( \chi_E^* = \chi_{[0,|E|]} \), and then

\[ \frac{1}{p} |E|^{1/p} \int_0^\infty t^{1/p-1} \chi_{[0,|E|]}(t) \, dt = \frac{1}{p} |E|^{1/p} |E|^{1/p} = 1. \]

Anyway, we won’t prove this here. Details can be found in [5].

Next we introduce a different definition of Lorentz space and prove it’s equivalent to definition 1. This definition has the advantage of making interpolation proofs easier (at the cost of proving the equivalence of the two definitions, of course).

**Definition 2.** A function \( f \) is in \( L^{p,q} \) if it can be decomposed as

\[ f = \sum_j c_j 2^{-j/p} f_j, \]

where \( \{c_j\}_j \in \ell^q \), and the functions \( f_j \)'s have supports disjoint from each other and the properties \( |f_j| \leq 1 \) and \( |\text{Supp} f_j| \leq 2^j \).

Notice that by this definition we get for free that \( L^{p,q_1} \subset L^{p,q_2} \) when \( q_1 \leq q_2 \), because of the corresponding property of the \( \ell^q \) spaces.
Proof. We prove definition 1 implies 2 first. Consider the expression for the quasi-norm $\|f\|_{L^{p,q}}$ and write
\[
\int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t} \sim \sum_{j \in \mathbb{Z}} 2^{j(q/p-1)} \int_{2^j}^{2^{j+1}} f^*(t)^q dt.
\]

Now put this aside for a moment and consider the expression $\int_a^b f^*(s) ds$. By the so-called layer cake representation formula one has
\[
\int_a^b f^*(s) ds = \int_{\{f^*(a) > |f| > f^*(b)\}} |f| dx + |\{ |f| = f^*(a) \}| + |\{ |f| = f^*(b) \}|
\]
notice though that we can always ignore the last two terms on the RHS by assuming that $|\{ |f| = \alpha \}| = 0$ for every $\alpha$, and we can always do so. Going back to the above expression then we have, since $(f^*)^q = (f^q)^*$,
\[
\int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t} \sim \sum_{j \in \mathbb{Z}} 2^{j(q/p-1)} \int_{\{f^*(2^j) > |f| > f^*(2^{j+1})\}} |f|^q dt,
\]
and we define $\Omega_j := \{ f^*(2^j) > |f| > f^*(2^{j+1}) \}$. Notice $f = \sum_j f \chi_{\Omega_j}$. Then, since $|\{ x : |f| > t \}| = |\{ s : f^*(s) > t \}|$, one also has $|\{ f^*(a) > |f| > f^*(b) \}| = |\{ s : f^*(a) > f^*(s) > f^*(b) \}| = b - a$, if our assumption that $|\{ |f| = \alpha \}| = 0$ holds; then
\[
|\Omega_j| = 2^{j+1} - 2^j = 2^j
\]
(when not empty - something that happens only if $f^*(2^j) = 0$). Therefore
\[
\sum_{j \in \mathbb{Z}} 2^{j(q/p-1)} \int_{\Omega_j} |f|^q dt \geq \sum_{j \in \mathbb{Z}} 2^{j(q/p-1)} |\Omega_j| f^*(2^{j+1})^q = \sum_{j \in \mathbb{Z}} 2^{jq/p} f^*(2^{j+1})^q,
\]
which implies that $\{2^{j/p} f^*(2^j)\}_{j \in \ell^q}$. Then we can set $c_j := 2^{j/p} f^*(2^j)$, and then write
\[
f = \sum_j c_j 2^{-j/p} f \chi_{\Omega_j} / f^*(2^j).
\]

More precisely, for any given $\varepsilon > 0$ there exists a function $g$ such that $f \leq g \leq (1+\varepsilon)f$ a.e. on $X$ and such that $|\{ |g| = \alpha \}| = 0 \forall \alpha$. For a proof of this easy statement see [1], which also helps getting used to rearrangements.
and $f_j := \frac{f_{\Omega_j}}{\|f_{\Omega_j}\|_{L^p(\Omega_j)}}$ has exactly the properties we ask for.

Now we prove the opposite, namely that 2 implies 1. Suppose first that $q < p$. Again

$$
\|f\|_{L^{p,q}}^q \sim \sum_k 2^{k(q/p-1)} \int_{\Omega_k} \left| \sum_j c_j 2^{-j/p} f_j \right|^q \, dx,
$$

where $\Omega_k$ is defined as before. By the disjointness of supports of $f_j$’s the RHS is

$$
\sum_{j,k} 2^{k(q/p-1)} 2^{-jq/p} |c_j|^q \int_{\Omega_k} |f_j|^q \, dx,
$$

and by the properties of $f_j$ the integral is bounded by $\min\{2^k, 2^j\}$. We then sum in $k$ first, optimizing accordingly: we get two terms, the first one being

$$
\sum_j 2^{-jq/p} |c_j|^q \sum_{k \leq j} 2^{k(q/p-1)} 2^k \sim \sum_j 2^{-jq/p} |c_j|^q 2^j 2^{jq/p} = \|c\|_{\ell^q},
$$

and the second one being

$$
\sum_j 2^{-jq/p} |c_j|^q 2^j \sum_{k > j} 2^{k(q/p-1)} \sim \sum_j 2^{-jq/p} |c_j|^q 2^j 2^{jq/p} = \|c\|_{\ell^q},
$$

and thus we’ve proved $\|f\|_{L^{p,q}} \lesssim \|c\|_{\ell^q}$. As for the case $q > p$, we proceed by duality, since then $q' < p'$. Thus we have to verify that

$$
\sup_{g: \|g\|_{L^{p',q}} \leq 1} |\langle f, g \rangle| < \infty,
$$

and by the previous point this amounts to prove that for all $g = \sum_k d_k 2^{-k/p'} g_k$ with $\|d\|_{\ell^{p'}} \lesssim 1$ one has

$$
|\langle f, g \rangle| \lesssim 1
$$

(with constant depending on $f$, of course). Then decompose $f$ as before and reduce to estimate

$$
\sum_{j,k} 2^{-j/p} 2^{-k/p'} |c_j| |d_k| \int |f_j g_k| \leq \sum_{j,k} 2^{-j/p} 2^{-k/p'} |c_j| |d_k| \min\{2^k, 2^j\} ;
$$

since $\min\{2^k, 2^j\} \leq 2^{\theta k + (1-\theta)j}$ for any $0 \leq \theta \leq 1$, we can optimize choosing a particular $\theta$ for $k - j > 0$ s.t. $-\frac{j}{p} - \frac{k}{p'} + \theta k + (1-\theta)j = (k-j) \left(\theta - \frac{1}{p'}\right) < 0$
and a different $\theta$ for $k - j < 0$, in such a way that the last sum is bounded by
\[ \sum_{j,k} 2^{-\delta |k-j|} |c_j||d_k| = \sum_{i,j} 2^{-\delta |i|} |c_j||d_{i+j}| \leq \sum_i 2^{-\delta |i|} \|c\|_{\ell^q} \|d\|_{\ell^{q'}} \lesssim \|c\|_{\ell^q}, \]

which proves the claim. (The case $p = q$ is left as an easy exercise). \(\square\)

Notice we have effectively proved that
\[ \|f\|_{L^{p,q}} \sim \inf \|c\|_{\ell^q}, \]
where the infimum is taken over all the possible decompositions of $f$ such as in definition 2.

### 0.1 Interpolation of Lorentz spaces

**Theorem 2** (Upgrading uniform restricted weak type to strong type). Let $T$ be of restricted weak type $(p,p)$ for all $p$ in a neighbourhood of $p_0$, with uniform constant. Then $T$ is of strong type $(p_0,p_0)$.

**Proof.** We proceed by duality. By the decomposition lemma and the observation at the end of its proof, it suffices to prove
\[ \left\| \left( \sum_j c_j 2^{-j/p_0} T f_j, \sum_k d_k 2^{-k/p_0'} g_k \right) \right\|_{\ell^{p_0}} \|d\|_{\ell^{p_0'}} \lesssim \|c\|_{\ell^{p_0}} \|d\|_{\ell^{p_0'}} \]

for any two functions $f \in L^{p_0}$, $g \in L^{p_0'}$, which can be worsened to
\[ \sum_{j,k} 2^{-j/p_0} 2^{-k/p_0'} |c_j||d_k| \|T f_j, g_k\| \lesssim \|c\|_{\ell^{p_0}} \|d\|_{\ell^{p_0'}}. \]

Now, observe that $f_j \in L^{p_1}$. Indeed, since $|\text{Supp} f_j| \leq 2^j$ then $f^*(2^j) = 0$, and since $|f_j| \leq 1$ also $f^* \leq 1$, and therefore
\[ \|f_j\|_{L^{p_1}} \leq \int_0^{2^j} t^{1/p} \frac{dt}{t} \sim 2^{j/p}; \]

same holds for $g_k$, namely $\|g_k\|_{L^{p_1}} \lesssim 2^{k/p'}$. Therefore, by assumption (and duality) it is
\[ |\langle T f_j, g_k \rangle| \lesssim 2^{j/p} 2^{k/p'} \]
with constant uniform for $p$ in a neighbourhood of $p_0$. Then our expression above on the LHS is bounded by

$$\sum_{j,k} 2^{-j/p_0} 2^{-k/p'_0} |c_j| |d_k| \inf_p 2^{j/p_2k/p'},$$

where the infimum is taken in the aforementioned neighbourhood of $p_0$. This sum looks like one we estimated before, and we optimize again the term $2^{-j/p_0} 2^{-k/p'_0} 2^{j/p_2k/p'}$.

Since $-\frac{j}{p_0} - \frac{k}{p'_0} + \frac{j}{p} = (k-j) \left( \frac{1}{p_0} - \frac{1}{p} \right)$, we choose $p$ bigger or smaller than $p_0$ according to the sign of $k-j$; then again the sum is bounded by

$$\sum_{j,k} 2^{-\delta k-j} |c_j| |d_k| = \sum_{i,j} 2^{-\delta |i|} |c_j| |d_k| \leq \sum_i 2^{-\delta |i|} \|c\|_{\ell^p} \|d\|_{\ell^{p'}} \lesssim \|c\|_{\ell^p} \|d\|_{\ell^{p'}},$$

which is exactly what we wanted. \hfill \square

Therefore we can bootstrap local restricted weak type $(p,p)$ estimates to strong $(p,p)$ estimates in the same range. An evident restriction of this is that $q = p$, so we ask whether we can interpolate between restricted weak type $(p,q)$ estimates. The answer is provided by the following

**Proposition 2.** Assume $1 < p_0, p_1, q_0, q_1 < \infty$. If a linear operator $T$ is restricted weak type $(p_0, q_0)$ and restricted weak type $(p_1, q_1)$, then it is also of restricted weak type $(p_0, q_0)$ for any $0 \leq \theta \leq 1$, where

$$\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$$

and

$$\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}.$$

**Proof.** The proof is immediate: again by duality one has to prove

$$|\langle T f, g \rangle| \lesssim \|c\|_{\ell^p} \|d\|_{\ell^q}$$

for $f = \sum_j c_j 2^{-j/p_0} f_j \in L^{p_0,1}$, $g = \sum_k d_k 2^{-k/q_0} g_k \in L^{q_0,1}$. But then

$$|\langle T f, g \rangle| \leq \sum_{j,k} 2^{-j/p_0} 2^{-k/q_0} |c_j| |d_k| \|\langle T f_j, g_k \rangle\|$$
\[ \leq \sum_{j,k} 2^{-j/p} 2^{-k/q} |c_j| |d_k| \min\{2^{j/p} 2^k/q, 2^{j/p} 2^k/q_0\} \]

\[ \leq \sum_{j,k} 2^{-j/p} 2^{-k/q} |c_j| |d_k| 2^{j/p} 2^k/q_0 \]

\[ = \sum_{j,k} |c_j| |d_k| \leq \|c\|_{\ell^1} \|d\|_{\ell^1}. \]

The last two proofs are taken from Tao’s notes in [4].

References


[5] G. S. de Souza, *A proof of Carleson’s theorem based on a new characterization of the Lorentz spaces \(L^{p,1}\) for \(1 < p < \infty\) and other applications."