A large part of modern harmonic analysis is concerned with understanding cancellation phenomena happening between different contributions to a sum or integral. Loosely speaking, we want to know how much better we can do than if we had taken absolute values everywhere. A prototypical example of this is the oscillatory integral of the form
\[ \int e^{i\phi(x)}\psi(x)\,dx \]
(solutions of dispersive PDEs often take this form). Here \( \psi \), called the amplitude, is usually understood to be “slowly varying” with respect to the real-valued \( \phi \), called the phase, and thus the oscillatory behaviour is given mainly by the complex exponential \( e^{i\phi(x)} \), where \( \phi \) denotes a parameter or list of parameters and generally \( \phi \) gets larger as \( \lambda \) grows, for example \( \phi(x) = \lambda \phi(x) \). Expressions of this form arise quite naturally in several problems, as we will see in Section 1, and typically one seeks to provide an upperbound on the absolute value of the integral above in terms of the parameters \( \lambda \). Intuitively, as \( \lambda \) gets larger the phase \( \phi \) changes faster and therefore \( e^{i\phi(x)} \) oscillates faster, producing more cancellation between the contributions of different intervals to the integral. We expect then the integral to decay as \( \lambda \) grows larger, and usually seek upperbounds of the form \( |\lambda|^{-\alpha} \). Notice that if you take absolute values inside the integral above you just obtain \( \|\psi\|_{L^1} \), a bound that does not decay in \( \lambda \) at all.

The main tool we will use is simply integration by parts. In the exercises you will also use a little basic complex analysis to obtain more precise information on certain special oscillatory integrals.

**Notation:** We will use \( A \lesssim B \) or \( A = O(B) \) to denote the estimate \( A \leq CB \) where \( C > 0 \) is some absolute constant, and \( A \sim B \) to denote the fact that \( A \lesssim B \lesssim A \). If the constant \( C \) depends on a list of parameters \( L \) we will write \( A \lesssim L \) or \( A = O_L(B) \).

We will further denote by \( C_k^L \) the space of \( k \)-times differentiable functions with compact support.

### 1. Motivation

In this section we shall showcase the appearance of oscillatory integrals in analysis with a couple of examples. You can find other interesting examples in the exercises, see particularly Exercise 8, Exercise 14 part viii), Exercise 15 and Exercise 21.

#### 1.1. Fourier transform of radial functions

Let \( f : \mathbb{R}^d \to \mathbb{C} \) be a radially symmetric function, that is there exists a function \( f_0 : \mathbb{R} \to \mathbb{C} \) such that \( f(x) = f_0(|x|) \) for every \( x \in \mathbb{R}^d \). Let’s suppose for simplicity that \( f \in L^1(\mathbb{R}^d) \) (equivalently, that \( f_0 \in L^1(\mathbb{R}, r^{d-1}\,dr) \)), so that it has a well-defined Fourier transform. It is easy to see\(^1\) that \( \hat{f} \) must also be radially symmetric, that is there must exist \( g : \mathbb{R} \to \mathbb{C} \) such that \( \hat{f}(\xi) = g(|\xi|) \); we want to understand its relationship with \( f_0 \). Therefore

\[ \int e^{i\phi(x)}\psi(x)\,dx \]

\(^1\)By composing \( f \) with a rotation and using a change of variable in the integral defining \( \hat{f} \).
we write using polar coordinates
\[ \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \xi} dx \]
\[ = \int_0^\infty \int_{S^{d-1}} f_0(r) e^{-2\pi i r \omega \cdot \xi} r^{d-1} d\sigma_{d-1}(\omega) dr, \]
\[ = \int_0^\infty f_0(r) r^{d-1} \left( \int_{S^{d-1}} e^{-2\pi i r \omega \cdot \xi} d\sigma_{d-1}(\omega) \right) dr \]
where \( d\sigma_{d-1} \) denotes the surface measure on the unit \((d-1)\)-dimensional sphere \( S^{d-1} \) induced by the Lebesgue measure on the ambient space \( \mathbb{R}^d \). By inspection, we see that the integral in brackets above is radially symmetric in \( \xi \), and so if we define
\[ J(t) := \int_{S^{d-1}} e^{-2\pi i t \omega \cdot e_1} d\sigma_{d-1}(\omega), \]
with \( e_1 = (1, 0, \ldots, 0) \), we have
\[ \hat{f}(\xi) = g(|\xi|) = \int_0^\infty f_0(r) r^{d-1} J(r|\xi|) dr. \quad (1.1) \]
This is the relationship we were looking for: it allows one to calculate the Fourier transform of \( f \) directly from the radial information \( f_0 \). You might also notice that \( J \) is simply the Fourier transform of the spherical measure \( d\sigma_{d-1} \).

Now we claim that the function \( J \) is an example of oscillatory integral of the type mentioned at the beginning. Indeed, observe that the inner product \( \omega \cdot e_1 \) depends only on the first component of \( \omega \); thus we write \( \omega = (\cos \theta, \omega' \sin \theta) \), with \( \theta \) the angle between \( \omega \) and \( e_1 \) and \( \omega' \in S^{d-2} \). By factorising the spherical measure \( d\sigma_{d-1} \) along the \( e_1 \) axis, we can use this change of variables to write
\[ J(t) = \int_{S^{d-2}} \int_0^\pi e^{-2\pi i t \omega' \cos \theta} (\sin \theta)^{d-2} d\theta d\sigma_{d-2}(\omega') \]
\[ = c_{d-2} \int_0^\pi e^{-2\pi i t \omega' \cos \theta} (\sin \theta)^{d-2} d\theta, \]
because the integrand does not depend on \( \omega' \) (here \( c_{d-2} = \int_{S^{d-2}} d\sigma_{d-2}(\omega') = \sigma_{d-2}(S^{d-2}) \)). It is now trivial to match the last expression to the one for an oscillatory integral where the parameter is \( t \). Later on we will see how to estimate it in terms of \( |t| \) and show that \( J(t) \) is bounded by a constant multiple of \((1 + |t|)^{-(d-1)/2} \).

The information we get from how fast \( J(t) \) decays in terms of \( |t| \) tells us of something interesting. Indeed, recall that if \( f \in L^1(\mathbb{R}^d) \) then its Fourier transform \( \hat{f} \) is continuous. In general, when \( 1 < p \leq 2 \) and \( f \in L^p(\mathbb{R}^d) \), we know by Hausdorff-Young inequality that \( \hat{f} \) is in \( L^p(\mathbb{R}^d) \), but we don’t know whether it is continuous or not. You can easily see this is generally not the case when \( p = 2 \). However, for radial functions one can show continuity of the Fourier transform if the exponent \( p \) is sufficiently close to \( 1 \). In particular

**Proposition 1.1.** If \( 1 \leq p < \frac{2d}{d+1} \) then the Fourier transform \( \hat{f} \) of any radial function \( f \in L^p(\mathbb{R}^d) \) is continuous away from 0.

This follows purely from the decay of the oscillatory integral \( J \) and you will prove it in Exercise [1].

1.2. Counting the number of ways in which a number can be written as the sum of two squares. Consider the following diophantine-type problem: when can we write an integer \( n \) as the sum of two squares, \( n = x^2 + y^2 \)? If you
fiddle around with examples a bit you might find that certain numbers have such a representation, say
\[ 5641 = 75^2 + 4^2, \]
while others, such as the successor 5642, have no such representation. Studying this problem more in depth you might discover the Fibonacci identity
\[ (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2, \]
and deduce as a consequence that it suffices to study the case where \( n \) is prime. If you went further down this road, you might learn that there is a way to write \( p \) prime as \( x^2 + y^2 \) if and only if \( p \equiv 1 \mod 4 \), and this gives the full answer for the problem: \( n = x^2 + y^2 \) has a solution if and only if in the prime factorisation of \( n \) the prime factors that are congruent to 3 mod 4 appear an even number of times each.

However, after this, you might notice that some numbers have even more than one representation as a sum of two squares: for example, \[ 5645 = 74^2 + 13^2 = 67^2 + 34^2. \]
The next step would then be to try and count how many distinct solutions there are. It turns out that this can be done in terms of the exponents in the prime factorisation of \( n \) (and the answer involves the Gaussian integers \( \mathbb{Z}[i] \)); however, this is not too convenient, because factorising large integers is knowingly hard. Let us take a different, less algebraic approach instead.

We can generalise the question a little by considering sums of more than two squares. Letting \( k \in \mathbb{N} \) be fixed and given an integer \( n \in \mathbb{N} \), we define the \( k \)-th sum-of-squares function to be
\[ r_k(n) := \# \{(x_1, \ldots, x_k) \in \mathbb{Z}^k \text{ such that } x_1^2 + \cdots + x_k^2 = n\}; \]
in words, \( r_k(n) \) is the number of ways in which \( n \) can be expressed as the sum of \( k \) squares. Then we want to study the behaviour of the function \( r_k(n) \) as \( n \) grows. In the following we will consider only \( k = 2 \).

If you plot the graph of \( r_2 \) you will quickly realise it is not a very regular function. A natural approach to deal with this irregularity is to study the averaged sequence instead, hoping it will behave better. This should help us answer the question “in how many ways can we expect to be able to write an arbitrary number \( n \) as a sum of two squares on average?”. In formulas, we want to study the behaviour of \( \sum_{n=1}^{N} r_2(n)/N \) as \( N \) tends to \( \infty \). The limit of the expression above is easy to find: the key is to notice that \( n = x^2 + y^2 \) means that the point \((x, y) \in \mathbb{Z}^2\) belongs to the circle of radius \( n^{1/2} \) centred at \((0, 0)\), and therefore to the ball of radius \( N^{1/2} \) with the same centre. Therefore
\[ \sum_{n=1}^{N} r_2(n) = \# \{(x, y) \in \mathbb{Z}^2 \text{ such that } x^2 + y^2 \leq N\}, \]
and a simple geometric argument shows that the limit of \( \frac{1}{N} \sum_{n=1}^{N} r_2(n) \) is \( \pi \) (see Exercise 2). The geometric argument actually says something more: namely, we can also give an upper bound on the error, that is
\[ \sum_{n=1}^{N} r_2(n) = \pi N + O(N^{1/2}). \]
The upper bound on the error is significant because it is of smaller order than the main term (which grows like \( O(N) \)). The limit of the averaged sequence was very easy to find, so now we ask a more sophisticated question: what is the behaviour

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2 Sequence A004018 in OEIS, [https://oeis.org/A004018/graph](https://oeis.org/A004018/graph)
of the error term? How much can the sum deviate from \( \pi N \)? \( O(N^{1/2}) \) is what one would expect from a uniformly random behaviour, but it turns out that it is far from optimal, and we can do better. However, we do not know yet how much better we can do! It is still an open problem from optimal, and we can do better. However, we do not know yet how much better we can do! It is still an open problem from optimal, and we can do better. However, we do not know yet how much better

In the range\( \epsilon > 0 \) the error \( \mathcal{E}(N) \) is bounded by \( C N^{1/4+\epsilon} \) for some constant \( C_\epsilon > 0 \). The estimate is certainly false with \( \epsilon = 0 \).

Proposition 1.2. For \( N \) large enough, we have the error term estimate

\[
|\mathcal{E}(N)| = O(N^{1/3}).
\]

If you stop for a moment to think about it, it is almost absurd that techniques developed to study oscillatory integrals can say something highly non-trivial about questions concerning the integers.

Proof. The idea is to use the Poisson Summation formula to reveal some interesting oscillation hidden in the problem. In the end, the proof will again rely on a good decay estimate for a certain oscillatory integral related to \( J \).

Recall that, for \( f \in S(\mathbb{R}^2) \), that is a Schwartz function, the Poisson Summation formula says that

\[
\sum_{n \in \mathbb{Z}^2} f(n) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n).
\]

The quantity we are interested in is \( \Theta(N) := \sum_{n \in \mathbb{Z}^2} 1_{B(0,N^{1/2})}(n) \), but \( 1_{B(0,N^{1/2})} \) (the characteristic function of a ball of radius \( N^{1/2} \)) is not in \( S(\mathbb{R}^2) \) - thus we have to regularise it. Let \( \delta > 0 \) be a small parameter to be chosen later and let \( \varphi \) be a bump function in \( C^\infty \), supported in the unit ball and such that \( \int_{\mathbb{R}^2} \varphi(x) dx = 1 \); we let \( \varphi_\delta \) denote the rescaled function \( \delta^{-2} \varphi(\delta^{-1} x) \) (so that \( \int \varphi_\delta = 1 \) too). Then we define

\[
\chi_{N,\delta} := 1_{B(0,N^{1/2})} * \varphi_\delta,
\]

which is in \( S(\mathbb{R}^2) \) and is an approximation to \( 1_{B(0,N^{1/2})} \). Applying the Poisson Summation formula to this function, we obtain

\[
\Theta_\delta(N) := \sum_{n \in \mathbb{Z}^2} \chi_{N,\delta}(n) = \hat{1}_{B(0,N^{1/2})}(0) \hat{\varphi}(0) + \sum_{n \neq 0 \in \mathbb{Z}^2} \hat{1}_{B(0,N^{1/2})}(n) \hat{\varphi}(\delta n);
\]

the first term is easily evaluated to be exactly \( \pi N \), our main term! Thus the second term is an error we have to control. Observe that, since \( 1_{B(0,N^{1/2})} \) is radial, we have by \( [1.1] \) that

\[
\hat{1}_{B(0,N^{1/2})}(n) = \int_0^{N^{1/2}} r J(r |n|) dr.
\]

The function \( J \) is not only an oscillatory integral, but it is itself oscillating, and with the techniques of the next section you will be able to show in Exercise \( 9 \) that

\[
\int_0^R r J(r) dr = O(R^{1/2}). \tag{1.2}
\]

After a change of variables, estimate \( [1.2] \) shows that \( |\hat{1}_{B(0,N^{1/2})}(n)| \lesssim N^{1/4} |n|^{-3/2} \). In the range \(|n| < \delta^{-1} \) we have \( |\hat{\varphi}(\delta n)| \lesssim 1 \) and therefore the above estimate gives us

\[
\sum_{|n| < \delta^{-1}, n \neq 0} |\hat{1}_{B(0,N^{1/2})}(n) \hat{\varphi}(\delta n)| \lesssim N^{1/4} \sum_{|n| < \delta^{-1}, n \neq 0} |n|^{-3/2} \sim N^{1/4} \delta^{-3/2}.
\]
In the range $|n| \geq \delta^{-1}$ things are even better, since $\hat{\varphi}$ decays fast, being in $C^\infty$; in particular, we have $|\hat{\varphi}(\delta n)| \lesssim (\delta |n|)^{-1}$ and thus, by the above same argument, this range also contributes $O(N^{1/4}\delta^{-1/2})$. Summarising, we have proven that

$$\Theta_\delta(N) = \pi N + O(N^{1/4}\delta^{-1/2}).$$

Now, observe that

$$\Theta_\delta((N^{1/2} - \delta)^2) \leq \Theta(N) \leq \Theta_\delta((N^{1/2} + \delta)^2);$$

this is because, as you can easily see by expanding the convolution,

$$\chi((N^{1/2} - \delta)^2, \delta) \leq 1\chi(0, N^{1/2}) \leq \chi((N^{1/2} + \delta)^2, \delta).$$

Therefore we have, thanks to (1.3) and the fact that $(N^{1/2} \pm \delta)^2 \approx N \pm 2N^{1/2}\delta$,

$$\Theta(N) - \pi N = \mathcal{E}(N) = O(N^{1/4}\delta^{-1/2}) + O(N^{1/2}\delta),$$

and if we optimize by choosing $\delta = N^{-1/6}$ we see that both terms above are $O(N^{1/3})$ and we are done.

2. Oscillatory integrals in one variable

In this section we present some techniques that allow one to estimate oscillatory integrals when one has a lowerbound for some of the derivatives of the phase. We will analyse objects of the form

$$I(\lambda) := \int_a^b e^{i\lambda \phi(x)}dx$$

and more in general

$$I_\psi(\lambda) := \int_a^b e^{i\lambda \phi(x)}\psi(x)dx.$$ 

Before we start, let’s consider the following heuristic. Suppose that we have functions $f, g$ and we know that $g$ is “slowly varying” - that is, the derivative $g'$ is small. In order to find a good upperbound for an expression of the form $|\int_a^b f(t)g(t)dt|$, we could take advantage of the fact that $g'$ is small by using integration by parts to estimate instead the expression $-\int_a^b F(t)g'(t)dt + \text{(boundary terms)}$, where $F$ is a primitive of $f$. What we gain is that now we are working with an integral for which we know one of the factors of the integrand, $g'$, is small. We will use this idea and variations thereof over and over.

2.1. Principle of non-stationary phase. We begin by considering the following case. Let $\psi \in C^\infty(\mathbb{R})$ have compact support in $(a, b)$ (so that in particular $\psi(a) = \psi(b) = 0$) and suppose that the phase $\phi \in C^\infty$ satisfies $\phi'(t) \neq 0$ for all $t \in [a, b]$. We claim that in this case the integral $I_\psi(\lambda)$ decreases very fast in $\lambda$, in particular

**Proposition 2.1** (Principle of non-stationary phase). Let $\psi, \phi$ be as above, that is $\psi \in C_{c}^\infty((a, b))$ and $\phi \in C^\infty$ is such that $\phi' \neq 0$ on $[a, b]$. Then for every $N > 0$ we have

$$|I_\psi(\lambda)| \lesssim_{N, \psi, \phi} |\lambda|^{-N}.$$ 

**Remark 1.** Notice that the bound given by the proposition above is only interesting when $|\lambda|$ is large. Indeed, when $|\lambda| \leq 1$ we can simply bound $|I_\psi(\lambda)| \leq \|\psi\|_{L^1} \lesssim 1$ by taking the absolute value inside the integral.

**Proof.** The proof is a simple integration-by-parts argument.

We want to use integration by parts a number of times. Notice that $(e^{i\lambda \phi})' = i\lambda \phi' e^{i\lambda \phi}$, so if we define the differential operator $D$ by

$$Df(t) := \frac{1}{i\phi'(t)} \frac{df}{dt},$$


we have $\lambda^{-1} D(e^{i\lambda \phi}) = e^{i\lambda \phi}$. Notice $D$ is well-defined because $\phi' \neq 0$. Using integration by parts we then have
\[
\int_a^b e^{i\lambda \phi(t)} \psi(t) dt = \int_a^b \lambda^{-1} D(e^{i\lambda \phi(t)}) \psi(t) dt \\
= e^{i\lambda \phi(b)} - e^{i\lambda \phi(a)} + \int_a^b e^{i\lambda \phi(t)} \lambda^{-1} D^t \psi(t) dt \\
= \lambda^{-1} \int_a^b e^{i\lambda \phi(t)} D^t \psi(t) dt,
\]
where the boundary term vanishes by the hypothesis on the support of $\psi$; here $D^t$ denotes the transpose of the operator $D$, namely $D^t f(t) = -\frac{1}{i} \frac{d}{dt} \left( \frac{f}{\phi'} \right)$. By repeating the argument $N$ times we get
\[
I_\psi(\lambda) = \lambda^{-N} \int_a^b e^{i\lambda \phi} (D^t)^N (\psi) dt,
\]
and we conclude by taking absolute values (the resulting integral is finite). □

We can interpret the principle of non-stationary phase as saying that, for a generic phase $\phi$, the behaviour of the oscillatory integrals $I_\psi(\lambda)$ is determined by the points where $|\phi'(t)| > 1$ for all $t \in (a, b)$. We make a further important assumption, that is we also assume that $\phi'$ is monotonic. Then we have

**Proposition 2.2.** If $\phi$ is such that $\phi'$ is monotonic and $|\phi'| > 1$ on $[a, b]$, we have
\[
|I(\lambda)| \leq C|\lambda|^{-1}
\]
for an absolute constant $C > 0$.

Some observations are in order, before we proceed to the proof:

i) First of all, the assumption that $\phi' \neq 0$ is fundamental: indeed, the statement where $\phi' = 0$ is false otherwise! (prove this in Exercise 4, see also Exercise 5.)

ii) Secondly, we cannot get a decay in $\lambda$ better than $|\lambda|^{-1}$, as the example $\phi(t) = t$ shows (you are cordially invited to do the calculation).

iii) Thirdly, by a simple rescaling of the phase, we can see that the proposition is more general than it looks: if the lowerbound on $|\phi'|$ becomes more generally $|\phi'| > \mu$, then the estimate becomes $|I(\lambda)| \leq C(\mu|\lambda|)^{-1}$.

iv) Finally, notice that the constant $C$ in the statement above depends neither on the phase $\phi$ nor on the interval $[a, b]$! In particular, if we allowed the constant to depend arbitrarily on the interval, we would trivially have that $|I(\lambda)| \leq |a - b|$, which is rather uninteresting.

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4Formally, the operator such that $\langle D f, g \rangle = \langle f, D^t g \rangle$ for all $f, g \in C^\infty_c$.

5Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is monotonic non-decreasing if $x < y \Rightarrow f(x) \leq f(y)$, and monotonic non-increasing if $x < y \Rightarrow f(x) \geq f(y)$. A function is monotonic if it is either monotonic non-decreasing or monotonic non-increasing.
Observations ii)-iv) are actually related, thanks to the scaling behaviour of the inequality. Indeed, if we ask for which values of \( \alpha \) we can have the inequality
\[
|I(\lambda)| \leq C_\phi |\lambda|^{-\alpha}
\]
hold with a constant \( C_\phi \) that is independent of \((a,b)\), then the answer is that necessarily it must be \( \alpha = 1 \). Prove this in Exercise 6.

Now we proceed with the proof.

Proof. We repeat the same integration-by-parts argument as before, except this time the boundary terms do not vanish. Thus we have, with \( D \) the same differential operator as before, that
\[
\int_a^b e^{i\lambda \phi(t)} dt = \left( \frac{e^{i\lambda \phi(b)}}{i \lambda \phi'(b)} - \frac{e^{i\lambda \phi(a)}}{i \lambda \phi'(a)} \right) + \int_a^b e^{i\lambda \phi(t)} \lambda^{-1} D^1 (1) dt.
\]
Since \( |\phi'| > 1 \), the boundary term is bounded by 2/|\lambda| simply by triangle inequality. As for the other term, \( \lambda^{-1} D^1(1) = -(i\lambda)^{-1} d/dt(1/\phi') \), and by taking absolute values inside the integral we have that it is bounded by
\[
\frac{1}{|\lambda|} \int_a^b \left| \frac{d}{dt} \left( \frac{1}{\phi'} \right) \right| dt.
\]
However, \( \phi' \) is monotonic and therefore so is \( 1/\phi' \); this means that in the last integral the derivative is single-signed, and therefore we can take the absolute value outside (something normally prohibited!) and obtain that it equals
\[
\frac{1}{|\lambda|} \int_a^b \left| d \left( \frac{1}{\phi'} \right) \right| dt.
\]
By the Fundamental Theorem of Calculus this is equal to \( |\lambda|^{-1} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \), which is bounded by \( 1/|\lambda| \), thanks to the monotonicity assumption. Thus we have proven the theorem with \( C = 3 \). \( \square \)

The natural step after this is to investigate what happens when the condition \( |\phi'| > 1 \) is violated but we still have lowerbounds for some higher derivatives. Indeed, we have

**Theorem 2.3 (Van der Corput’s lemma).** Let \( k \geq 2 \) and let \( \phi \in C^k \) be such that \( |\phi^{(k)}| > 1 \) on \((a,b)\). Then
\[
|I(\lambda)| \leq C_k |\lambda|^{-1/k},
\]
where \( C_k \) is an absolute constant depending only on \( k \).

Notice that \( |\lambda|^{-1/k} \) decays slower than \( |\lambda|^{-1} \) as \( \lambda \to \infty \). This is to be expected, since the phase “slows down” near the zeros of \( \phi' \) and hence there is less overall cancellation. It is indeed sharp, as the example \( \phi(t) = t^k \) over \([0,1]\) shows (again, you are cordially invited to do the calculation - you can use complex integration and regularize the integral by introducing a factor of \( e^{-\varepsilon t^k} \), then take \( \varepsilon \to 0 \) at the end; see also Exercise 15).

Again, it goes without saying that if we have \( |\phi^{(k)}| > \mu \) instead, then the inequality becomes \( |I(\lambda)| \leq C_k (\mu |\lambda|)^{-1/k} \). Moreover, the exponent \( 1/k \) is necessary for the inequality to hold with \( C_k \) independent of the interval \((a,b)\) (see again Exercise 6).

---

6As the subscript indicates, the constant might depend on \( \phi \) in principle. However, this will not be the case for us, at least for now.
Remark 2. You should have noticed that this time we are not making an explicit monotonicity assumption such as in Proposition 2.2. However, we implicitly still have that $|\phi^{(k)}| > 1$ for some $k \geq 2$ implies that $\phi^{(k-1)}$ attains value zero in at most one point of $(a, b)$, and thus $\phi^{(k-2)}$ attains value zero in at most two points, and so on; iterating, we see that $\phi'$ has at most $k - 2$ changes of monotonicity, or in other words we can partition $(a, b)$ into at most $k - 1$ intervals such that $\phi'$ is monotonic on each interval.

Proof. The proof proceeds by induction on $k$. The case $k = 1$ (with the additional assumption that $\phi'$ be monotonic) has been proven in Proposition 2.2. Notice that if $|\phi''| > 1$ then $\phi'$ is monotonic.

Assume now that the statement is true for $k - 1$. Since $|\phi^{(k)}| > 1$, the $(k - 1)$-th derivative $\phi^{(k-1)}$ can have at most one zero in $(a, b)$. Denote this zero by $t_0$ and split the integral as

$$
\int_{t_0-\delta}^{t_0+\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^{b}
$$

for some $\delta > 0$ to be chosen later. In the interval $(a, t_0 - \delta)$ the function $\phi^{(k-1)}$ is never zero, but moreover we have by the assumption on its derivative $\phi^{(k)}$ that $|\phi^{(k-1)}| > \delta$; similarly on the interval $(t_0 + \delta, b)$. By inductive hypothesis we therefore have that

$$
\left| \int_{t_0-\delta}^{t_0+\delta} e^{i\lambda \phi(t)} dt \right| + \left| \int_{t_0+\delta}^{b} e^{i\lambda \phi(t)} dt \right| \leq 2C_{k-1}(\delta |\lambda|)^{-1/(k-1)}.
$$

As for the remaining integral, we just estimate trivially

$$
\left| \int_{t_0-\delta}^{t_0+\delta} e^{i\lambda \phi(t)} dt \right| \leq 2\delta.
$$

Putting everything together, we have shown that $|I(\lambda)| \leq 2C_{k-1}(\delta |\lambda|)^{-1/(k-1)} + 2\delta$, and by choosing $\delta = |\lambda|^{-1/k}$ this gives

$$
|I(\lambda)| \leq (2C_{k-1} + 2)|\lambda|^{-1/k},
$$

proving the induction (with $C_k = 2C_{k-1} + 2$). \qed

Although the constant $C_k$ obtained with the above argument suffices for most (if not all) applications, it’s interesting to notice it is not optimal in $k$. See Exercise 11 if you are interested in determining the correct behaviour of the optimal constant in $k$.

It is a simple matter of integrating by parts to extend Van der Corput’s lemma to the oscillatory integrals $I_\psi(\lambda)$ as well. We obtain

**Corollary 2.4.** Let $\psi \in C^1$ and let the phase $\phi$ satisfy $|\phi^{(k)}| > 1$ in $(a, b)$ for some $k \geq 1$ (if $k = 1$, assume additionally that $\phi'$ is monotonic). Then the inequality

$$
|I_\psi(\lambda)| \leq C'_k \left[ |\psi(b)| + \int_a^b |\psi'(t)| dt \right] \cdot |\lambda|^{-1/k}
$$

holds, with $C'_k > 0$ an absolute constant depending only on $k$.

Thus the overall constant will depend on $a, b$ through the amplitude $\psi$ (though notice that if we assume $\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}$ is finite, it is independent of $a, b$), but we have isolated its dependence in the term in square brackets. The constant is still independent of the phase $\phi$.

**Proof.** Let $\Phi(t) := \int_a^t e^{i\lambda \phi(s)} ds$, so that

$$
I_\psi(\lambda) = \int_a^b \Phi'(t) \psi(t) dt.
$$
Integrating by parts and taking absolute values we have that $|I_\psi(\lambda)|$ is bounded by

$$|\Phi(b)\psi(b)| + \int_a^b |\Phi(t)||\psi'(t)|\,dt.$$  

For the first term, by Theorem 2.3 we can estimate $|\Phi(b)| \lesssim \lambda^{-1/k}$. Similarly, the second term can then be estimated by $C_k|\lambda|^{-1/k} \int_a^b |\psi'(t)|\,dt$. Summing these contributions gives the stated bound. □

Notice the integration-by-parts trick we used here is different from the one used for Propositions 2.1, 2.2.

2.3. Method of stationary phase. Now we go back to the observation made at the end of Section 2.1. Recall that the portions of the integral $I_\psi(\lambda)$ where the phase satisfies $\phi' \neq 0$ contribute at most $O(\psi,N)$ for arbitrary $N > 0$ and thus can typically be treated as an error term. The behaviour of $I_\psi(\lambda)$ is therefore determined by the zeros of $\phi'$ and higher derivatives of the phase. In particular, one can perform a full asymptotic expansion of the oscillatory integral $I_\psi(\lambda)$. This is known as the method of stationary phase; it’s particularly useful in physics, where it prominently appears in the semi-classical approximation to Quantum Field Theory.

We can state (one version of) the method as follows.

**Theorem 2.5** (Method of stationary phase). Assume that $\phi \in C^\infty$. Let $k \geq 2$ and assume that

$$\phi'(t_0) = \ldots = \phi^{(k-1)}(t_0) = 0 \quad \text{and that } \phi^{(k)}(t_0) \neq 0.$$  

If $\psi \in C^\infty_c$ is supported in a sufficiently small neighbourhood of $t_0$, then there exist coefficients $a_j$ for $j \in \mathbb{N}$ (each depending only on finitely many derivatives of $\phi$ and $\psi$ at $t_0$) such that

$$I_\psi(\lambda) \simeq e^{i\lambda \phi(t_0)} \lambda^{-1/k} \sum_{j \in \mathbb{N}} a_j \lambda^{-j/k}, \quad (2.1)$$

where by $\simeq$ we mean that for all $n > 0$ we have for the sum truncated at $n - 1$ that

$$\left| I_\psi(\lambda) - e^{i\lambda \phi(t_0)} \lambda^{-1/k} \sum_{j=0}^{n-1} a_j \lambda^{-j/k} \right| \lesssim \psi,\phi,n |\lambda|^{-n/k}$$

as $\lambda \to \infty$, and moreover for all $n, \ell > 0$ we have

$$\left| \left( \frac{d}{d\lambda} \right)^\ell \left[ I_\psi(\lambda) - e^{i\lambda \phi(t_0)} \lambda^{-1/k} \sum_{j=0}^{n-1} a_j \lambda^{-j/k} \right] \right| \lesssim \psi,\phi,\ell,n |\lambda|^{-\ell-n/k}.$$  

The coefficients $a_j$ can be determined explicitly - for example, when $k = 2$ an explicit calculation of $a_0$ shows that the main term in $I_\psi(\lambda)$ is

$$e^{i\lambda \phi(t_0)} \left( \frac{2\pi}{-i\lambda \phi''(t_0)} \right)^{1/2} \psi(t_0).$$

This is excellent for a quick-and-dirty estimate of complicated integrals. We will not prove the above theorem here since it would make these notes unnecessarily long, but if you fancy you can prove it yourself in Exercise 13 following the guidelines provided there.
3. Oscillatory integrals in several variables

In the previous section we have analysed the situation for single variable phases, that is for integrals over (intervals of) $\mathbb{R}$. In this section, we will instead study the higher dimensional situation, when the phase is a function of several variables. Things are unfortunately generally not as nice as in the single variable case, as you will see.

In order to avoid having to worry about connected open sets of $\mathbb{R}^d$ (see Exercise 18 for the sort of issues that arise in trying to deal with general open sets of $\mathbb{R}^d$), in this section we will study mainly objects of the form

$$I_\psi(\lambda) := \int_{\mathbb{R}^d} e^{i\lambda u(x)} \psi(x) dx,$$

where $\psi$ has compact support. We have switched to $u$ for the phase to remind the reader of the fact that it is a function of several variables now.

3.1. Principle of non-stationary phase - several variables. The principle of non-stationary phase we saw in Section 2.1 continues to hold in the several variables case.

Given a phase $u$, we say that $x_0$ is a critical point of $u$ if

$$\nabla u(x_0) = (0, \ldots, 0).$$

Proposition 3.1 (Principle of non-stationary phase - several variables). Let $\psi \in C^\infty_c(\mathbb{R}^d)$ (that is, smooth and compactly supported) and let the phase $u \in C^\infty$ be such that $u$ does not have critical points in the support of $\psi$. Then for any $N > 0$ we have

$$|I_\psi(\lambda)| \lesssim N, \psi, u |\lambda|^{-N}.$$

Proof. The proof is simply a reduction to the single variable case of Proposition 2.1. Indeed, by assumption there exists a $c > 0$ such that $|\nabla u(x)| > 2c$ for $x$ in the support of $\psi$, and therefore for any such $x$ there exists a small ball $B_x$, centered at $x$ and a unit vector $\xi_x$ such that $\xi_x \cdot \nabla u(y) > c$ for every $y \in B_x$. By compactness we can find a finite collection of such balls, $\{B_j\}_j$, that covers the support of $\psi$; with a partition of unity associated to this collection, we can therefore write $\psi = \sum_j \psi_j$ with $\psi_j$ supported in $B_j$. Since the number of balls is finite (depending only on $\psi, u$), it suffices to prove the claimed bound for a single $I_{\psi_j}(\lambda)$. By a change of coordinates, we can assume $\xi_j = (1, 0, \ldots, 0)$ and since

$$\int_{\mathbb{R}^d} e^{i\lambda u(x)} \psi_j(x) dx = \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} e^{i\lambda u(x_1, x_2, \ldots, x_d)} \psi_j(x_1, x_2, \ldots, x_d) dx_1 \right) dx_2 \cdots dx_d,$$

we can conclude by applying Proposition 2.1 to the inner integral and then integrating in the remaining variables (over the projection of $B_j$).

3.2. Van der Corput’s lemma in several variables. Proceeding by analogy to what was done in Section 2 we now want to study what happens when we have a lower bound on some (possibly mixed) derivative of the phase $u$. We look for inequalities that share the same scaling behaviour as the single variable ones in Proposition 2.2 and Theorem 2.3.

We will denote by $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index, that is an element of $\mathbb{N}^d$, and by $|\alpha|$ the sum $\alpha_1 + \ldots + \alpha_d$. Then $\partial^\alpha$ denotes the partial derivative of order $|\alpha|

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}.$$

We have the following several-variables version of Van der Corput’s lemma (more precisely, of Corollary 2.4).
Theorem 3.2 (Van der Corput’s lemma in several variables). Let $\psi \in C^1_c(B(0, 1))$ (that is, compactly supported in the unit ball). Assume that for every $x$ in the support of $\psi$ the phase $u$ satisfies

$$|\partial^\alpha u(x)| > 1$$

for a certain fixed multi-index $\alpha$, with $|\alpha| \geq 1$. Furthermore we assume $u \in C^{[\alpha] + 1}$. Then we have

$$|\mathcal{I}_\psi(\lambda)| \leq C_{[\alpha], u} \left( \|\psi\|_{L^\infty} + \|\nabla \psi\|_{L^1} \right) \cdot |\lambda|^{-1/|\alpha|},$$

where the constant $C_{[\alpha], u}$ depends only on $|\alpha|$ and on the phase $u$ (in particular, on the derivatives of $u$ up to order $|\alpha| + 1$).

One can appreciate the superficial similarity between Corollary 2.4 and Theorem 3.2. We reiterate that if the hypothesis on the phase becomes

$$i) \text{ unfortunately, the constant now depends on the phase too, which can sometimes be an issue when one has to deal with multiple phases at once;}$$

$$ii) \text{ as you will see from the proof, the constant also depends on the size of the support of } \psi \text{ if one removes the extra hypothesis that } \psi \in C^1_c(B(0, 1)), \text{ and thus the inequality does not have the nice scaling properties that it had in the single variable case;}$$

$$iii) \text{ the estimate is no longer tight: indeed, if we take a phase like } \psi \text{ with each } \psi_j \text{ supported in one of such balls. It suffices to estimate } \mathcal{I}_\psi(\lambda) \text{ thus the inequality does not have the nice scaling properties that it had in the single variable case; }$$

$$\text{so that } |\partial_x \partial_y u| = 1, \text{ and an amplitude that is just a bump function supported near the origin, the theorem gives us a bound of } O(|\lambda|^{-1/2}), \text{ but in reality one can prove the better estimate } O(|\lambda|^{-1}) \text{ (see Section 3.3 or Exercise 19).}$$

Proof. The proof will again work by reducing to the single variable case (and this is the source of the inefficiencies mentioned above).

Let $k = |\alpha|$. In order to reduce to the single variable case, we need to show that the assumption $|\partial^\alpha u| > 1$ implies a similar lowerbound for some derivative of the form $(\xi \cdot \nabla)^k u$ - that is, a $k$-th derivative in a fixed direction, $(\xi \cdot \nabla) f(x) = d\xi(f(x + \xi))$, so that we can apply single variable results. Indeed, the real vector space of partial derivatives of order $k$ admits a basis of the above form: there exist unit vectors $\xi^{(1)}, \ldots, \xi^{(m)}$ (where $m = \# \{ \beta \in \mathbb{N}^d \text{ s.t. } |\beta| = k \}$) such that any partial derivative of order $k$ can be expressed as a linear combination of derivatives $(\xi^{(j)} \cdot \nabla)^k$. Prove this in Exercise 19.

Since $\partial^\alpha u$ can be expressed as a linear combination of $(\xi^{(j)} \cdot \nabla)^k$, for each $x$ in the support of $\psi$ there exists a unit vector $\xi_x$ such that $|((\xi_x \cdot \nabla)^k u(x))| \geq 1$; moreover, since we have assumed that $u \in C^{k+1}$, we have that $\|u\|_{C^{k+1}}$ is bounded on the support of $\psi$, and therefore for a small ball $B_x$ (of radius $\lesssim \|u\|_{C^{k+1}}^{-1}$) centered at $x$ we have actually

$$|((\xi_x \cdot \nabla)^k u(y))| \geq 1 \quad \text{for all } y \in B_x$$

(with a worse constant than the one implicit in the previous lowerbound). As in the proof of Proposition 3.1 we can find a finite covering of the support of $\psi$ by such balls and use them to create a partition of unity that allows us to split $\psi$ as $\sum_j \psi_j$ with each $\psi_j$ supported in one of such balls. It suffices to estimate $\mathcal{I}_\psi(\lambda)$ separately ($\psi$ being supported in $B(0, 1)$, the number of such balls only depends

\footnote{For a suggestive calculation: estimate \( \int_{\Omega} \exp(i\lambda xy) dxdy \) where $\Omega = \{(x, y) : |x| + |y| \leq 1\}$ (hint: rotate coordinates and use single variable Van der Corput’s lemma).}
on \(|u|_{C^{k+1}}\). After a change of variables, we can assume that \(\xi = (1,0,\ldots,0)\) for the ball under examination, and therefore we write again (with \(x' = (x_2,\ldots,x_d)\))
\[\int_{B(0,1)} e^{i\lambda u(x)} \psi_j(x) dx = \int_{\mathbb{R}^{d-1}\cap B(0,1)} \left( \int e^{i\lambda u(x_1,x')} \psi_j(x_1,x') dx_1 \right) dx'.\]
Applying Corollary 2.4 to the inner integral we get that it is bounded by
\[\lesssim_k \left[ \|\psi\|_{L^\infty} + \int |\partial_{x_1} \psi(x_1,x')| dx_1 \right] \cdot |\lambda|^{-1/k};\]
integrating in the remaining variables gives us the estimate we want.

3.3. Method of stationary phase in several variables - non-degenerate phases. When \(|\alpha| = 2\), we can obtain better estimates than the ones given by Theorem 2.5 provided that we have some extra information about the phase being well-behaved. In particular, if \(x_0\) is a critical point for phase \(u\), we say that it is a non-degenerate critical point if
\[\det(\text{Hess}(u)(x_0)) \neq 0,\]
where \(\text{Hess}(u)\) is the Hessian matrix \(\left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{ij}\).

If the phase has only non-degenerate critical points, we can assert an analogue of Theorem 2.5 for phases of several variables.

**Theorem 3.3.** Assume that \(\psi \in C_c^\infty(\mathbb{R}^d)\) and that the phase \(u \in C^\infty(\mathbb{R}^d)\) has \(x_0\) as a non-degenerate critical point. If \(\psi\) is supported in a sufficiently small neighbourhood of \(x_0\) (in particular, there are no other critical points of \(u\) in it), then there exist coefficients \(a_j\) for \(j \in \mathbb{N}\) (each depending only on finitely many derivatives of \(u\) and \(\psi\) at \(x_0\)) such that
\[I_{\psi}(\lambda) \asymp e^{i\lambda u(x_0)} \lambda^{-d/2} \sum_{j \in \mathbb{N}} a_j \lambda^{-j},\]
where \(\asymp\) is in the same sense as in Theorem 2.5. In particular, \(|I_{\psi}(\lambda)| \lesssim_{\phi, \psi} |\lambda|^{-d/2}.

When \(u(x,y) = xy\), we see that this gives \(I_{\phi}(\lambda) = O(\lambda^{-1})\) as claimed before.

We do not prove this theorem either (but if you have proven Theorem 2.5 in Exercise 15, you can prove this one as well in Exercise 20 following the same strategy with little extra effort), but rather make a remark.

**Remark 3.** The theorem above is limited to the case \(|\alpha| = 2\). Indeed, observe that in the single variable case the fact that \(\phi^{(k)}(t_0) \neq 0\) and \(\phi'(t_0) = \phi''(t_0) = \ldots = \phi^{(k-1)}(t_0) = 0\) implies that we can put the phase in the simple canonical form \(\phi(t) = t^k\) (after a change of variable); however, in several variables there is no simple analogue of this, except in the case \(|\alpha| = 2\), in which we can put the phase \(u\) in a standard quadratic form - hence our limitation.

3.4. Estimates independent of the phase. As observed above, one of the downsides of Van der Corput’s lemma in several variables is that the constant ends up depending on the phase \(u\) as well, which is sometimes a problem. It is also known that, if we consider the case \(\psi \equiv 1\) and thus look at oscillatory integrals of the form \(\int_\Omega e^{i\lambda u(x)} dx\) with \(\Omega\) a connected (bounded) open set, the constant cannot be made independent of \(\Omega\) (prove this in Exercise 15).

It is reasonable to look for estimates with constants that are independent of as many parameters as possible. The above observation forces us to give up independence in \(\Omega\), and therefore we concentrate on independence on the phase \(u\). It turns out that it is possible to obtain oscillatory integral estimates with constant independent of the phase, but it seems that one has to pay the price somehow (see the log in the
estimate below). Such estimates form a rich theory, but here we will only present an example to give you a taste of what kind of results and techniques are involved.

**Theorem 3.4.** If $u : [0, 1]^2 \to \mathbb{R}$ is such that

$$\left| \frac{\partial^2 u}{\partial x \partial y}(x, y) \right| > 1 \quad \forall (x, y) \in [0, 1]^2$$

and furthermore

$$\frac{\partial^3 u}{\partial x^2 \partial y}(x, y) \neq 0 \quad \forall (x, y) \in [0, 1]^2,$$

we have

$$\left| \int_{[0, 1]^2} e^{i\lambda u(x, y)} \, dx \, dy \right| \lesssim (\log(2 + |\lambda|))^{1/2} |\lambda|^{-1/2}.$$

In particular, the constant is independent of $u$.

Obviously, there is an analogous statement for $\partial_x \partial_y^2 u$ in place of $\partial_x^2 \partial_y u$. Notice that the assumption on this third mixed derivative is nothing more than a monotonicity assumption (which turns out to be necessary).

**Proof.** Assume $\lambda > 1$ for convenience. By Cauchy-Schwarz and Fubini, we have

$$\left| \int_{[0, 1]^2} e^{i\lambda u(x, y)} \, dx \, dy \right|^2 \leq \int_0^1 \left( \int_0^1 e^{i\lambda u(x, y)} \, dy \right)^2 \, dx \cdot \int_0^1 1 \, dx$$

$$= \int_0^1 \left( \int_0^1 e^{i\lambda (u(x, y) - u(x, y'))} \, dx \right) \, dy \, dy'.$$

Letting $\phi_{y, y'}(x) := u(x, y) - u(x, y')$ we have that

$$\phi_{y, y'}'(x) = \partial_x u(x, y) - \partial_x u(x, y') = \int_{y'}^y \partial_x \partial_y u(x, t) \, dt,$$

which by the hypotheses on $u$ implies that $\phi_{y, y'}'$ is monotonic (since $\phi_{y, y'}''$ is single-signed) and that $|\phi_{y, y'}'(x)| > |y - y'|$. By Proposition 2.2 it follows that

$$\left| \int_0^1 e^{i\lambda (u(x, y) - u(x, y'))} \, dx \right| \lesssim \min\{1, (\lambda|y - y'|)^{-1}\}.$$

Thus the above is bounded by

$$\int_0^1 \int_0^1 \min\{1, (\lambda|y - y'|)^{-1}\} \, dy \, dy'$$

$$= \lambda^{-1} \int_{|y - y'| > \lambda^{-1}} |y - y'|^{-1} \, dy \, dy' + \int_{|y - y'| < \lambda^{-1}} 1 \, dy \, dy' \lesssim \lambda^{-1} \log \lambda + \lambda^{-1},$$

which gives the claim. □

It is not known whether the estimate holds without the logarithmic term.
Exercises

This is a (long) list of exercises meant to improve your grasp on the topic and provide you with useful ideas for the future - do the ones you like. Those that require a bit more work are marked by ★’s - they are not necessarily harder, just longer maybe. The ones unmarked are probably more important for a basic understanding though. In any case, hints are given in the next section. Since the exercises are meant to be a complement to the lectures, don’t be afraid to have a look at the hints - you are actually encouraged to do so.

Exercise 1. (★) Let $p$ be such that $1 \leq p < 2d/(d+1)$. Show, by using the decay estimate $J(t) \lesssim (1+|t|)^{-(d-1)/2}$, that if $f$ is a radial function in $L^p(\mathbb{R}^d)$ and $\mathbb{R}^d \ni \xi \neq 0$, then the Fourier transform $\hat{f}$ is continuous at $\xi$. (See hints for a walk-through)

Exercise 2. Show, using the geometric interpretation given in Section 1.2 of solutions to $n = x^2 + y^2$ as points in a disk, that there exists some constant $C > 0$ such that for all $N > 1$

$$|\sum_{n=1}^{N} r_2(n) - \pi N| \leq CN^{1/2}$$

holds.

Exercise 3. If $D$ is a (linear) differential operator, its transpose is given by the linear operator $D^t$ that satisfies

$$(Df, g) = (f, D^t g)$$

for all test functions $f, g \in C^\infty_c$. While $D$ satisfies Leibniz’s rule

$$D(fg) = gDf + fDg,$$

show that $D^t$ in general does not.

Exercise 4. Show that Proposition 2.2 is false if the assumption that $\phi'$ is monotonic is dropped. In other words, untangling the quantifiers, construct a family of phases $\{\phi_\lambda\}_\lambda$ such that $|\phi_\lambda'| > 1$ on $(a, b)$ for each $\lambda$, but $|\lambda||I(\lambda; \phi_\lambda)|$ is not bounded as $\lambda \to +\infty$.

[To destroy cancellation in $I(\lambda)$, you should aim to make $\phi_\lambda'$ oscillate at the scale $|\lambda|^{-1}$. Indeed, consider the following: looking only at the real part of the integral, we are trying to make $\int_0^1 \cos(2\pi \lambda x) dx$ large (and positive, say). The function $\cos(2\pi \lambda x)$ is positive for $x$ in $[-\frac{1}{14\pi}, \frac{1}{14\pi}] + \frac{1}{2}\mathbb{Z}$, and negative otherwise. If you look at the graph of $\phi_\lambda$ (and I encourage you to make a drawing of the argument that follows), you want it to spend as much time as possible in the horizontal bands $\mathbb{R} \times \left(\left[-\frac{1}{14\pi}, \frac{1}{14\pi}\right] + \frac{1}{2}\mathbb{Z}\right)$ and as little time as possible in the complement, that is in the bands $\mathbb{R} \times \left(\left[\frac{1}{14\pi}, \frac{3}{14\pi}\right] + \frac{1}{2}\mathbb{Z}\right)$, so that the positive contribution outweighs the negative one. To achieve this, $\phi_\lambda''$ should be small when $\phi_\lambda$ is in the former bands (but not too small, since we still want $|\phi_\lambda'| > 1$), and quite large when in the latter bands. In particular, $\phi_\lambda'$ will not be monotone and it will oscillate between two behaviours over an interval of length $\sim \lambda^{-1}$.]}

Exercise 5. Show the following weaker version of Van der Corput’s lemma for $k = 1$: assume that $|\phi'| > \mu$ but, instead of assuming that $\phi'$ is monotonic, we only assume that $|\phi''| < M$ on $(a, b)$; prove that

$$|I(\lambda)| \lesssim \left(\frac{1}{\mu} + \frac{M(b-a)}{\mu^2}\right)|\lambda|^{-1}.$$
Exercise 6. Let $\phi$ be a phase that satisfies $|\phi^{(k)}(t)| > 1$ on $(a,b)$ for some $k \geq 1$. Show that a necessary condition for inequality

$$|I(\lambda)| \leq C_\phi |\lambda|^{-\alpha}$$

to hold with $C_\phi$ independent of $[a,b]$ is that $\alpha = 1/k$.

Exercise 7. Show, using integration by parts (as many times as necessary) and Van der Corput’s lemma, that we have the estimate

$$|J(t)| \lesssim d(1 + |t|)^{-(d-1)/2},$$

where $J$ is the function introduced in Section 1.

Exercise 8. The Airy equation is the dispersive PDE

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, \\ u(0,x) = f(x). \end{cases}$$

By taking a Fourier transform in the spatial variable $x$, it is not hard to see that the solution to the Airy equation can be written formally as the convolution

$$u(t,x) = \frac{1}{2\pi} \int f(y) \frac{1}{t^{1/3}} \text{Ai} \left( \frac{x-y}{t^{1/3}} \right) dy,$$

where $\text{Ai}(x)$ denotes the Airy function

$$\text{Ai}(x) := \int_{\mathbb{R}} e^{i(\xi^3 + x\xi)} d\xi,$$

which is clearly an oscillatory integral (provided the integral exists!). You will show that it satisfies the following estimates:

1) For $x > 1$ we have superpolynomial decay, that is for every $N > 0$ we have $|\text{Ai}(x)| \lesssim |x|^{-N}$.
2) For $-1 \leq x \leq 1$ we have $|\text{Ai}(x)| \lesssim 1$.
3) For $x < -1$ we have $|\text{Ai}(x)| \lesssim |x|^{-1/4}$.

Follow these steps:

i) It will be useful to smoothly split $\mathbb{R}$ dyadically. Let $\phi$ be a $C^\infty$ bump function supported in $[-2,2]$, with $\phi \equiv 1$ on $[-1,1]$; define $\psi(\xi) := \phi(\xi/2) - \phi(\xi)$, and $\psi_j(\xi) := \psi(2^{-j}\xi)$. Show that $\phi(\xi) + \sum_{j \in \mathbb{N}} \psi_j(\xi) = 1$ for every $\xi$ and that $\psi_j$ is supported in $2^j < |\xi| < 2^{j+2}$. We can make sense of the integral defining $\text{Ai}(x)$ as the limit as $n \to \infty$ of

$$\int e^{i(\xi^3 + x\xi)} \phi(\xi) d\xi + \sum_{j=0}^{n} \int e^{i(\xi^3 + x\xi)} \psi_j(\xi) d\xi.$$

ii) Splitting the integral using the decomposition above, show 1) by the principle of non-stationary phase. In doing this, you will also prove that the limit above indeed exists pointwise.

iii) Show 2) adapting the argument you just used for ii).

iv) Show 3) by Corollary 2.4 (and the splitting).

v) Show the simple dispersive estimate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R},dx)} \lesssim |t|^{-1/3} \|f\|_{L^1(\mathbb{R})}.$$
Exercise 9. Show estimate (1.2) using integration by parts, Van der Corput’s lemma and splitting the integrals where necessary; that is, when $d = 2$ (thus $J(r) = \int_0^\pi e^{-2\pi ir \cos \theta} d\theta$) show that we have for large $R$

$$\left| \int_0^R r J(r) dr \right| \lesssim R^{3/2}.$$ 

The point is that $\int_0^R r J(r) dr$ is itself an oscillatory integral (expanding $J$ and using Fubini shows this is the case).

[Notice this is a much better estimate than the one you would get from just plugging in the estimate $|J(r)| \lesssim |r|^{-1/2}$, which would give you the much larger $O(R^{3/2})$. There is therefore additional cancellation to be exploited.]

Exercise 10. An estimate of the form

$$|\{ t \in (a, b) \text{ s.t. } |\phi(t)| < \lambda \}| = O(\lambda^\alpha)$$

for some $\alpha > 0$ is called a sublevel-set estimate. Notice that it is only interesting for small $\lambda$. These estimates are in many ways related to oscillatory integral estimates, as this exercise will show:

i) Show that Van der Corput’s lemma implies that, under the same hypotheses on $\phi$, for every $\lambda > 0$ we have

$$|\{ t \in (a, b) \text{ s.t. } |\phi(t)| < \lambda \}| \lesssim_\lambda \lambda^{1/k}. \tag{♠}$$

We stress that the constant only depends on $k$.

ii) Show conversely that if we assume that (♠) holds, we can prove Van der Corput’s lemma by splitting the interval $(a, b)$ into $\{ t \in (a, b) \text{ s.t. } |\phi'(t)| < \theta \}$ and its complement, where $\theta$ is a parameter to be optimized (Remark 2 might come in handy).

iii) Now prove (♠) directly by induction in $k$ (thus providing another proof of Van der Corput’s lemma, when combined with ii)).

iv) (Only if you know about $p$-adics, otherwise ignore) Let $p$ be a prime and let $\mathbb{Z}_p$ be the ring of $p$-adic integers, with its non-archimedean valuation $|\cdot|_p$. If $P$ is a polynomial in $\mathbb{Z}_p[X]$, what does the sublevel-set

$$\{ x \in \mathbb{Z}_p \text{ s.t. } |P(x)|_p < p^{-s} \}$$

correspond to, in simpler terms? And what would an estimate like (♠) mean in this context?

Exercise 11. (⋆) The proof of Van der Corput’s lemma we have presented in these notes gave us a constant that is exponential in $k$; in particular, you can easily see that it gives $C_k \sim 2^k$ (Theorem 2.3). This behaviour is not sharp. Indeed, if $B_k$ denotes the smallest constant for which the inequality

$$|I(\lambda)| \leq B_k |\lambda|^{-1/k}$$

holds (under the hypotheses of Theorem 2.3), then the constant actually grows linearly in $k$, namely $B_k \sim k$.

In this exercise you will obtain the optimal behaviour of the constant, showing that $|I(\lambda)| \leq C_k |\lambda|^{-1/k}$. This will be achieved by using the same strategy as in ii)-iii) of Exercise 10 but with an improved dependency on $k$ for estimate (♠).

i) Show that, if $E$ is a measurable subset of $\mathbb{R}$ with $|E| > 0$, for any $k$ one can find $x_0, \ldots, x_k \in E$ such that for any $j \in \{0, 1, \ldots, k\}$

$$\prod_{i : i \neq j} |x_i - x_j| \geq \left( \frac{|E|}{2^k} \right)^k.$$  

[hint: squeeze $E$ into an interval of length $|E|$, then do the obvious thing.]
ii) Show the following generalisation of the mean-value theorem: let \( x_0 < x_1 < \cdots < x_k \) and \( f \in C^k([x_0, x_k]) \); then there exists \( y \in (x_0, x_k) \) such that

\[
f^{(k)}(y) = (-1)^k k! \sum_{j=0}^{k} \frac{f(x_j)}{\prod_{i:j \neq j}(x_i - x_j)}.
\]

[Hint: use Lagrange interpolation at the \( x_j \)'s to get a polynomial approximant, then apply Rolle's theorem to the difference \( k \) times.]

iii) Show that if \( \phi \) is such that \( \phi^{(k)}(t) > 1 \) for all \( t \in \mathbb{R} \), then we have the sublevel-set estimate (see Exercise 10)

\[
|\{ t \in \mathbb{R} \text{ s.t. } |\phi(t)| < \lambda \}| \leq (2e)((k+1)!)^{1/k} \lambda^{1/k}.
\]

In particular, use i) and ii) above applied to \( E = \{ t \in \mathbb{R} \text{ s.t. } |\phi(t)| < \lambda \} \) and \( f = \phi \).

iv) Use Stirling's approximation to show that \( ((k+1)!)^{1/k} \sim k \).

v) Prove Van der Corput's lemma using the same proof as in ii) of Exercise 10 (using the sublevel-set estimate above that has an improved constant) to conclude that

\[
|I(\lambda)| \leq Ck|\lambda|^{-1/k}.
\]

vi) Show that \( B_k \) cannot grow slower than \( k \) [check a canonical example and use complex integration techniques; don't expect a straightforward calculation though.]

Exercise 12. You might have noticed that the statement of Theorem 2.5 (the method of stationary phase) omits the case \( k = 1 \). Indeed, in this case there can be no expansion in terms of powers of \( \lambda^{-1} \), since the integral is \( O_N(\lambda^{-N}) \) for every \( N > 0 \) (by non-stationary phase principle). However, in the case where the support of \( \psi \) is not strictly contained in \( (a, b) \) (and thus non-stationary phase does not apply), a similar statement holds. Prove an asymptotic expansion of the form

\[
\lambda^{-1} \sum_{j \in \mathbb{N}} \left( a_j e^{i\lambda \phi(a)} + b_j e^{i\lambda \phi(b)} \right) \lambda^{-j}
\]

for \( I_\phi(\lambda) \), under the hypotheses that \( \phi' \neq 0 \) in \( (a, b) \) and that \( \phi, \psi \in C^\infty \) (in particular, in general \( \psi(a), \psi(b) \neq 0 \)). Calculate explicitly the first few coefficients \( a_j, b_j \).

[Hint: just use integration by parts repeatedly.]

Exercise 13. (★★) In this exercise you will prove the method of stationary phase as stated in Theorem 2.5. The proof will proceed in stages, and we can assume \( t_0 = 0 \) and \( \phi(0) = 0 \) for simplicity.

i) Begin with the case \( k = 2 \), which will already contain all the main ideas.

Take \( \phi(t) = t^2 \) and \( \psi(t) = t^m e^{-t^2} \) for some \( m > 0 \) (which is not compactly supported, but still very concentrated around the origin); let \( a = -\infty \) and \( b = +\infty \). Show, using standard complex integration techniques, that

\[
I_\phi(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda t^2} t^m e^{-t^2} dt
\]

equals (fixing the principal branch of \( z^{-(m+1)/2} \))

\[
(1 - i\lambda)^{-(m+1)/2} \int_{-\infty}^{+\infty} t^m e^{-t^2} dt,
\]

and argue by a power series expansion in \( \lambda^{-1} \) that therefore it satisfies (2.1).

ii) Now we keep the quadratic phase \( \phi(t) = t^2 \) but replace the gaussian factor \( e^{-t^2} \) directly with a compactly supported function \( \eta \). Thus, let \( \eta \in C_c^\infty \) and let \( \psi(t) = t^m \eta(t) \). Show, by splitting the region of integration smoothly into
one close to 0 and one away from 0 (according to a well chosen parameter),
that in this case
\[ |I_{\epsilon n}(\lambda)| \lesssim g_{n,m} \lambda^{-(m+1)/2}. \]
More precisely, let \( \varphi \) denote a smooth bump function supported in \([-1, 1]\) and decompose \( 1 = \varphi(t/\delta) + (1 - \varphi(t/\delta)) \). For the region close to 0 you can take the absolute value inside the integral and give a trivial estimate; for the region away from 0, apply the integration by parts argument as in the proof of Proposition 2.1 \( N \) times, where \( 2N \) is bigger than \( m + 1 \), then take the absolute value inside and estimate the size of this integral. Finally, optimise in the parameter \( \delta \).

iii) Show, using the above same arguments, that if \( g \in S(\mathbb{R}^d) \) (that is, \( g \) is a Schwartz function) is such that \( g \equiv 0 \) in a neighbourhood of 0, then for any \( N > 0 \) one has \( |I_g(\lambda)| = O_{N,g}(|\lambda|^{-N}) \). (The phase is still \( \phi(t) = t^2 \).)

iv) Now, still in the case \( \phi(t) = t^2 \), we tackle the general \( \psi \) case. Write
\[
\int e^{i\lambda t^2} \psi(t) dt = \int e^{i\lambda t^2} e^{-t^2} (e^{t^2} \psi(t)) \eta(t) dt,
\]
where \( \eta \in C^\infty_0 \) is such that \( \eta(t) = 1 \) for \( t \) in the support of \( \psi \). Perform a Taylor expansion of \( e^{t^2} \psi(t) \) to degree \( n \) and substitute into \( I_\psi(\lambda) \). Then use i)-ii)-iii) to argue that (2.1) holds for each term you get out of this procedure.

v) Now consider the general \( k = 2 \) case. The idea is to deform the phase to turn it into \( t^2 \) again. Find a diffeomorphism from a sufficiently small neighbourhood \( U \) of 0 that, by a change of variables, achieves this, and then conclude that (2.1) holds by iv).

vi) Generalise the above to \( k > 2 \). The substitute for i) will be the identity
\[
\int_0^\infty e^{i\lambda t^k} e^{-t^m} dt = C_{k,m}(1 - i\lambda)^{-(m+1)/k},
\]
which is similarly proven by standard complex integration techniques.

**Exercise 14.** In these notes we have only considered amplitudes \( \psi \) that are everywhere \( C^\infty \). In applications however (see viii) below) more singular cases arise, and it is instructive to see that some of those cases can still be treated using the techniques here developed. This exercise will moreover show you that sometimes some cancellation can also arise from the amplitude.

Let \( P(t) = \sum_{j=0}^d c_j t^j \) be a polynomial of degree \( d \). You will show that for arbitrary \( 0 < \epsilon < R \) it holds
\[
\left| \int_{|t| < |R|} e^{i\epsilon P(t)} \frac{dt}{t} \right| \leq C_d
\]
with a constant that depends only on \( d \) (and not on the coefficients of \( P! \)). Notice that the amplitude \( 1/t \) is only \( C^\infty \) away from 0, and has a bad singularity there. In the following, it will be really important that integration is over a set symmetric with respect to the origin such as \( \{ t \in \mathbb{R} : \epsilon < |t| < R \} = (-R, -\epsilon) \cup (\epsilon, R) \) above.

i) Show by a change of variable that we can assume that \( c_d = 1 \) (at the price of changing \( \epsilon, R \); but these are arbitrary).

ii) Show, using Corollary 2.4 that the portion of the integral over \( 1 < |t| < R \) is bounded by \( O_d(1) \), so that it suffices to concentrate on the remaining part \( \epsilon < |t| < 1 \). (Notice that even in the range \( |t| > 1 \) we cannot estimate the integral by taking absolute values, since the resulting quantity grows like \( \log R \); we really need oscillatory integral estimates!).

iii) Let \( P_k(t) := \sum_{j=1}^k c_j t^j \) (thus \( P_0 = P \)). Using the trivial estimate \( |e^{it} - 1| \lesssim |\theta| \) and taking absolute values, show that
\[
\left| \int_{\epsilon < |t| < 1} e^{iP(t)} \frac{dt}{t} - \int_{\epsilon < |t| < 1} e^{iP_{d-1}(t)} \frac{dt}{t} \right| \lesssim_d 1.
\]
iv) Use steps i)-iii) repeatedly to reduce to the case of integral \( \int_{|t|<1/\epsilon} e^{i\epsilon_1 t} \frac{dt}{t} \).

v) It remains to show that the above integral is \( O_d(1) \). Show that \( \int_{|t|<\epsilon} \frac{dt}{t} = 0 \) for any \( a, b > 0 \) (this is a form of cancellation!). Therefore, you can arbitrarily subtract 1 from the exponential value \( e^{i\epsilon_1 t} \) above...

vi) Split the integral as \( \int_{|t|<\epsilon_1/\epsilon_1} + \int_{|t|<|t|<1} \) and use again Corollary 2.4 to the second integral to show that it is \( O_d(1) \).

vii) It remains to estimate \( \int_{|t|<1/\epsilon_1} e^{i\epsilon_1 t} \frac{dt}{t} \). Using v), show that it is the same as \( \int_{|t|<1/\epsilon_1} (e^{i\epsilon_1 t} - 1) \frac{dt}{t} \) and estimate the latter by taking the absolute value and using the trivial estimate for \( |e^a - 1| \). This concludes the proof.

viii) Consider the operator

\[
f \mapsto \lim_{R \to 0, \epsilon \to 0} \int_{|t|<|t|<R} f(x - t, y - t^2) \frac{dt}{t};
\]

this is known as the Hilbert transform along a parabola. Show that this operator is \( L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) bounded. [hint: use Plancherel and take a Fourier transform of the operator above; after some calculations, you should recognize a certain object...]

Exercise 15. (★) The discrete counterparts to oscillatory integrals, which we haven’t discussed in these notes, are exponential sums, that is sums of the form

\[
\sum_{n=1}^{N} e^{2\pi if(n)}
\]

(sometimes they are normalised by dividing by \( N \)). Such sums are ubiquitous in number theory, where they are used to encode all sorts of information, from the number of solutions to diophantine equations to the behaviour of the Riemann Zeta function.

It should not be too surprising that some of the techniques developed for oscillatory integrals transfer over to the treatment of exponential sums. Indeed, integration by parts has a very direct discrete counterpart in summation by parts. In this exercise you will prove some estimates for exponential sums in the spirit of the ones studied in these lectures. We will use a notational convention from number theory, namely we introduce the function \( e(x) := e^{2\pi i x} \) so that \( e^{2\pi i f(n)} = e(f(n)) \).

You will prove an analogue of Proposition 2.2.

i) Show the trigonometric identity

\[
\frac{1}{1 - e^{i\theta}} = \frac{1}{2} + i \frac{\cot(\theta/2)}{2}.
\]

ii) Let \( g(n) = e(f(n))/e(f(n) - \pi f(n - 1)) \); show by summation by parts that

\[
\sum_{n=1}^{N} e(f(n)) = [e(f(N))g(N) - e(f(0))g(1)] - \sum_{n=1}^{N-1} e(f(n))(g(n+1) - g(n)).
\]

iii) Using i), show that

\[
g(n+1) - g(n) = (i/2) \left[ \cot(\pi f(n+1) - f(n)) - \cot(\pi f(n) - f(n-1)) \right].
\]

iv) So far we just did a bunch of algebra; now we start doing analysis. Assume that \( f' \) is monotonic in the interval \( [0, N] \) and that for some \( 1/2 > \delta > 0 \) we have \( \text{dist}(f'(x), Z) > \delta \) for all \( x \in [0, N] \) (that is, the derivative always stays within two consecutive integers, never getting any closer than \( \delta \) to any
of them). Show that the sequence \( n \mapsto f(n) - f(n - 1) \) is also monotonic and that for some fixed integer \( k \) we have for all \( n \in \{1, \ldots, N\} \)
\[
k + \delta \leq f(n) - f(n - 1) \leq k + 1 - \delta.
\]
v) Since \( \cot \) is also monotonic in intervals of the form \( [\pi k, \pi (k + 1)] \), show that under the hypotheses in iv) on \( f \) we have
\[
\left| \sum_{n=1}^{N-1} e(f(n))(g(n + 1) - g(n)) \right| \leq |g(N) - g(0)|,
\]
and deduce that the above is bounded by \( \lesssim \delta^{-1} \). [hint: monotonicity allows you to take the absolute values from inside of a sum to outside of it; also, \( \cot(x) \leq 1/x \) for \( x \in (0, \pi/2] \).]

vi) Put everything together (check those remaining terms) and conclude that you have shown: if \( f' \) is monotonic on interval \( I \) and \( \text{dist}(f'(x), \mathbb{Z}) > \delta \) for all \( x \in I \), then
\[
\left| \sum_{n \in I \cap \mathbb{Z}} e(f(n)) \right| \lesssim \delta^{-1}.
\]
(\( \diamondsuit_1 \))

Compare this with Proposition 2.2 and appreciate the similarities between the respective proofs.

Now we move on to the 2nd derivative. You will prove an analogue of Van der Corput’s lemma, that is: if \( f \in C^2 \) is such that for all \( x \in I \) we have
\[
\lambda < |f''(x)| < C\lambda
\]
(for some \( C > 1 \), then
\[
\left| \sum_{n \in I \cap \mathbb{Z}} e(f(n)) \right| \lesssim C|I|\lambda^{1/2} + \lambda^{-1/2}.
\]
(\( \diamondsuit_2 \))

The strategy will be to partition \( I \) into intervals where \( (\diamondsuit_1) \) above applies, and where it doesn’t to use trivial estimates, and finally optimize between the two.

a) Assume for convenience that \( f'' \) is positive and let \( I = [a, b] \). Show that the range of \( f' \) is \( J := [f'(a), f'(b)] \) and show that \( |J| \leq C\lambda|I| \).

b) Partition \( J \) into intervals close to integers and intervals that avoid integers. More precisely, let \( 1/2 > \delta > 0 \) be a parameter to be chosen and define
\[
A_k := [k - \delta, k + \delta] \cap J,
B_k := [k + \delta, k + 1 - \delta] \cap J.
\]
Show that \( A_k \) and \( B_k \) are not empty for at most \( O(C\lambda|I| + 1) \) values of \( k \) and that they partition \( J \).

c) Since \( f' \) is monotonic, we can partition \( I \) into a disjoint union of intervals \( (f')^{-1}(A_k) \) and \( (f')^{-1}(B_k) \). Show that on the first ones we have trivially
\[
\left| \sum_{n \in (f')^{-1}(A_k) \cap \mathbb{Z}} e(f(n)) \right| \lesssim \delta/\lambda
\]
and on the second ones by \( (\diamondsuit_1) \) we have
\[
\left| \sum_{n \in (f')^{-1}(B_k) \cap \mathbb{Z}} e(f(n)) \right| \lesssim \delta^{-1}.
\]

d) Now put everything together to show that we have
\[
\left| \sum_{n \in I \cap \mathbb{Z}} e(f(n)) \right| \lesssim (C\lambda|I| + 1) \left( \frac{\delta}{\lambda} + \frac{1}{\delta} \right),
\]
and conclude estimate (22) by choosing an optimal $\delta$ (i.e. minimise $\delta/\lambda + 1/\delta$).

Again, you should appreciate the similarities between this proof and the proof of Van der Corput’s lemma we have given in the notes.

e) An estimate is only interesting if it beats the trivial one, which in this case is $|\sum_{n \in \mathbb{Z}} e(f(n))| \leq |I|$ (by triangle inequality; here $|I| > 1$). For which range of values of $\lambda$ is estimate (22) interesting?

f) Consider truncations of the Riemann Zeta function along the critical line, that is sums of the form $\sum_{n=A}^{B} n^{-1/2-\varepsilon}$. Argue that in order to bound such sums, by summation by parts one can reduce to study sums of the form $\sum_{n=B'}^{D'} n^{-\varepsilon}$ instead. As $n^{-\varepsilon} = e(-t/2\pi) log(n)$, verify that the estimates (22), (23) that you proved above are well suited to treat these last oscillatory sums. This is how bounds on the growth of $|\zeta(1/2 + it)|$ as $t \to \infty$ are usually proven.

**Exercise 16.** Let $V$ denote the real vector space of partial derivatives of order $k$, that is the space of differential operators of the form

$$\sum_{\alpha : |\alpha| = k} c_\alpha \partial^\alpha.$$  

Show that $V$ admits a basis of vectors all of the form $(\xi \cdot \nabla)^k$, where the $\xi$’s are unit vectors.

*Equivalently*, you can identify $V$ with the real vector space of homogeneous polynomials of degree $k$ in $d$ variables.

*[hint: Using the identification with homogeneous polynomials, it suffices to find an inner product on $V$ such that if a polynomial is orthogonal to all polynomials of the form $(\xi \cdot X)^k$, with $X = (X_1, \ldots, X_d)$, then it has to be the zero polynomial.]*

**Exercise 17.** Show again the estimate

$$|J(t)| \lesssim_d (1 + |t|)^{-d+1/2}$$

as in Exercise 7 but this time using the method of stationary phase in several variables as in Theorem 2.3. Use the expression for $J$ in terms of an integral over the sphere, then express the half-sphere as the graph of a function.

Relatively, recall that if $\mu$ is a finite measure on $\mathbb{R}^d$, we can define its Fourier transform $\hat{d\mu}$ as

$$\hat{d\mu}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x).$$

Calculate the Fourier transform $\hat{d\sigma}(\xi)$ of the spherical measure $d\sigma$ on the $(d-1)$-dimensional sphere $S^{d-1}$ and deduce its decay properties from the above. Oscillatory integrals provide a great way to obtain such estimates.

Finally, consider the flat surface in $\mathbb{R}^d$ given by $[-1,1]^{d-1} \times \{0\}$ and let $d\mu$ be its surface measure. Show that there is a direction in which $\hat{d\mu}$ does not decay at all. This should convince you that the source of the extra cancellation that makes $\hat{d\sigma}$ decay is the curvature of the underlying surface.

**Exercise 18.** In this exercise you will show that in dimension greater than 1 it is impossible to have estimates of the form

$$\left| \int_{\Omega} e^{i \lambda u(x)} dx \right| \leq C|\lambda|^{-1/|\alpha|}$$

with constant $C$ independent of $\Omega$ (here we are assuming $|\partial^\alpha u| > 1$).

i) Repeat the proof you gave for i) of Exercise 10 to show that the inequality above would imply

$$\{x \in \Omega \text{ s.t. } |u(x)| < \lambda \} \leq C'|\lambda|^{1/|\alpha|}$$
with \( C' \) independent of \( \Omega \). Thus it will suffice to disprove the sublevel-set estimates.

ii) First we show that if \( \Omega \) is not bounded, there is no hope to have an estimate like the above. Consider \( \Omega = [-R, R]^2 \) and \( u(x, y) = \frac{1}{2} (x + y)^2 \). We have \( \partial_x \partial_y u = 1 \). Show that, however,
\[
|\{(x, y) \in \Omega \text{ s.t. } |u(x, y)| < \lambda\}| \sim R\lambda^{1/2},
\]
so there is an unavoidable dependence on \( \text{diam}(\Omega) \).

iii) Now we show that even if we take \( \Omega \) contained in a bounded set (say \([0, 1]^2\)), the constant will still depend on properties of \( \Omega \). Here the trick will be to consider a set \( \Omega \) that looks a bit like a comb with \( N \) teeth. In particular, define
\[
\Omega_j := [1/2, 1] \times [j/N, (j + 1/2)/N]
\]
for \( j = 0, \ldots, N - 1 \) (this will be the \( j \)-th “tooth”) and
\[
\Omega := [0, 1/2] \times [0, 1] \cup \bigcup_{j=0}^{N-1} \Omega_j,
\]
We define the phase \( u \) piecewise. Let \( \varphi \) be a smooth positive function such that \( \varphi \equiv 0 \) on \([0, 5/8] \) and \( \varphi \equiv 1 \) on \([3/4, 1] \). Then on each tooth we prescribe \( u \) to behave the same way: for \((x, y) \in \Omega_j\), \( u(x, y) = y - j\varphi(x)/N \). On the rest of the comb, we prescribe: for \((x, y) \in [0, 1/2] \times [0, 1]\), \( u(x, y) = y \). Show that \( |\partial_{\nu} u| = 1 \) on \( \Omega \) and show that for \( \lambda \) sufficiently smaller than \( N^{-1} \) we have
\[
|\{(x, y) \in \Omega \text{ s.t. } |u(x, y)| < \lambda\}| \gtrsim N\lambda,
\]
so that the constant necessarily depends on \( \Omega \).

Exercise 19. (★) In this exercise you will prove a result that is weaker than the full asymptotics provided in Theorem 3.3 but which is sufficient for most applications. In particular, under the same hypotheses (in particular, the non-degeneracy of the critical point of \( u \)), you will prove the estimate
\[
|\mathcal{I}_\psi(\lambda)| \lesssim_{u, \psi, d} |\lambda|^{-d/2}.
\]
The proof is quite similar to that of Theorem 3.4. For starters, let \( u, \psi \) satisfy the hypotheses of Theorem 3.3. We expand \( |\mathcal{I}_\psi(\lambda)|^2 \) into a double integral in \( dx \, dy \) and change variables to \((y, z)\) by introducing \( z = x - y \). We obtain
\[
|\mathcal{I}_\psi(\lambda)|^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda(u(y+z)-u(y))} \psi(y+z)\psi(y)dydz.
\]
i) Verify the calculations above.

ii) The expression we have is of the form \( \int_{\mathbb{R}^d} \mathcal{J}_{\lambda}(z)dz \) with \( \mathcal{J}_{\lambda} \) an oscillatory integral. You will show that for any \( N > 0 \) it is \( |\mathcal{J}_{\lambda}(z)| \lesssim_N (1 + \lambda|z|)^{-N} \). Show that this will imply that \( |\mathcal{I}_\psi(\lambda)|^2 \lesssim |\lambda|^{-d} \), hence concluding the estimate we want.

iii) Now you will prove the claimed \( |\mathcal{J}_{\lambda}(z)| \lesssim_N (1 + \lambda|z|)^{-N} \) using essentially the same proof given for Proposition 2.1. The oscillatory term of \( \mathcal{J}_{\lambda}(z) \) is \( e^{i\lambda(u(y+z)-u(y))} \); calculate its \( \nabla_y \) gradient and use this information to craft a differential operator \( D \) such that \( D(e^{i\lambda(u(y+z)-u(y))}) = e^{i\lambda(u(y+z)-u(y))} \).

iv) Your differential operator should have the form \( D = (i\lambda)^{-1}\theta_{\lambda}(y) \cdot \nabla_y \) for some explicit \( \mathbb{R}^d \)-valued function \( \theta_{\lambda}(y) \). Show by simple Taylor expansion arguments that for any multi-index \( \alpha \) one has
\[
\|\partial_{\theta}^\alpha \theta_{\lambda}(y)\| \lesssim_{\alpha} |z|^{-1}.
\]
Exercise 20. (★★) In this exercise you will prove the method of stationary phase in several variables, that is Theorem 3.3, following the same strategy used in Exercise 13 (which you should attempt before attempting this one). Assume $x_0 = 0$.

Let $x = (x_1, \ldots, x_d)$, and if $\alpha \in \mathbb{N}^d$ is a multi-index let $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

i) Assume the phase is purely a non-degenerate quadratic form $Q(x) = \sum_{j=1}^d x_j^2 - \sum_{j=\ell+1}^d x_j^2$. Take $\psi(x) = e^{-\|x\|^2/10}$ and show that \((3.1)\) holds, that is
\[
\int_{\mathbb{R}^d} e^{i\lambda Q(x)} e^{-\|x\|^2/10} \, dx \simeq \lambda^{-d/2 - |\alpha|/2} \sum_{j \in \mathbb{N}} \tilde{a}_{j,\alpha} \lambda^{-j}.
\]
[hint: the integral factorises, then you can repeat what you did in Exercise 13.]

Now we need to prove a result for a compactly supported $\psi$. Indeed, let $\eta(x)$ a smooth function compactly supported around the origin and let $\psi(x) = x^\alpha \eta(x)$. You will show that, with the same purely quadratic phase $Q$ as above, we have
\[
|I_\psi(\lambda)| \lesssim_{\alpha, \eta, Q} |\lambda|^{-(d+|\alpha|)/2}.
\]

To do so, we will need to use integration by parts at some point with respect to some $\partial_{x_k}$, and thus it will be convenient to localise in that direction. We achieve this as follows: we roughly divide $\mathbb{R}^d$ into the cones
\[
\Gamma_k := \{ x \in \mathbb{R}^d \text{ s.t. } \|x\|^2/d \leq |x_k|^2 \}
\]
for $k \in \{1, \ldots, d\}$; you can see that $\Gamma_k$ is a cone with axis the $x_k$-axis. Show that
\[
\mathbb{R}^d = \bigcup_{k=1}^d \Gamma_k.
\]

Using slightly larger cones
\[
\tilde{\Gamma}_k := \{ x \in \mathbb{R}^d \text{ s.t. } \|x\|^2/(2d) \leq |x_k|^2 \},
\]
argue by a partition of unity that it suffices to reduce to the case where the amplitude is $\psi(x) \cdot G_k(x)$ with $G_k$ a function homogeneous of degree 0 (that is, $G_k(x) = G_k(x/\|x\|)$, smooth away from the origin, supported in $\tilde{\Gamma}_k$ and identically 1 in a conical neighbourhood of the $x_k$-axis (say, points $x$ such that $(9/10) \|x\|^2 \leq |x_k|^2$).

iii) This step will be the exact analogue of step ii) in Exercise 13. With phase $Q$ and amplitude $x^\alpha \eta(x) G_k(x)$, split the region of integration smoothly into $\|x\| \lesssim \delta$ and the complement. For the first part use a trivial estimate, and for the second part use repeated integration by parts with respect to the differential operator
\[
D_k f(x) := \pm \frac{1}{2i \lambda x_k} \partial_{x_k}.
\]

Optimizing in $\delta$, conclude the estimate in ii). [hint: notice that $D_k$ applied to $e^{i\lambda Q}$ returns exactly $e^{i\lambda Q}$]
iv) Using the same arguments as above, show that if \( g \in \mathcal{S}(\mathbb{R}^d) \) is identically 0 near the origin, then for any \( N > 0 \)
\[
|Z_g(\lambda)| \lesssim_{N,g} |\lambda|^{-N}. 
\]

v) For a generic amplitude \( \psi \) and phase \( Q \) as above, write
\[
\int e^{i\lambda Q(x)} \psi(x) dx = \int e^{i\lambda Q(x)} e^{-||x||^2} (e^{||x||^2} \psi(x)) \eta(x) dx
\]
with \( \eta \) compactly supported and identically 1 on the support of \( \psi \). Perform a Taylor expansion of \( e^{||x||^2} \psi(x) \) to some finite degree, then use i)-iv) above to argue that (3.1) holds for this oscillatory integral.

vi) Finally, if \( u \) is a generic phase satisfying the hypotheses of Theorem 3.3 argue by Morse’s Lemma that there is a smooth change of variables that lets us put \( u \) in the form \( Q \) above (provided \( \psi \) has small enough support). Therefore, conclude by v). [If you do not know about Morse’s lemma, it says more or less precisely what we claimed; look it up, it is a very convenient tool.]

**Exercise 21.** (★★★) Oscillatory integrals are prototype solutions for many partial differential equations:

1) Show that the oscillatory integral
\[
u(x,t) := \int e^{it(\xi^2 + x \cdot \xi)} \psi(\xi) d\xi
\]
is a solution for the Schrödinger equation \( i\partial_t u - \Delta u = 0 \).

2) Show that the oscillatory integral
\[
u(x,t) := \int e^{it(\xi^2 + x \cdot \xi)} \psi(\xi) d\xi
\]
is a solution for the wave equation \( \partial_t^2 u - \Delta u = 0 \).

3) Show that the oscillatory integral
\[
u(x,t) := \int e^{it(\xi^2 + x \cdot \xi)} \psi(\xi) d\xi,
\]
where \( \langle \xi \rangle \) (the so-called japanese bracket) denotes \( (1 + |\xi|^2)^{1/2} \), is a solution for the Klein-Gordon equation \( \partial_t^2 u - \Delta u + m^2 u = 0 \) \((m > 0 \) is a constant - the mass).

Now consider the wave equation in \( \mathbb{R}^d \) above. You will prove some decay estimates for a packet of waves of frequency approximately \( 2^k \).

Let \( \psi(\xi) \) be a \( C^\infty \) function supported in the annulus \( 1 < |\xi| < 2 \) and radially symmetric, for simplicity. Fix an integer \( k \in \mathbb{Z} \) and consider the solution to the wave equation
\[
\Phi_k(x,t) := \int e^{i(x \cdot \xi + t\langle \xi \rangle)} \psi \left( \frac{\xi}{2^k} \right) d\xi,
\]
which is frequency localised in \( 2^k < |\xi| < 2^{k+1} \). You will show that for any \( N > 0 \) the above solution obeys the estimate
\[
|\Phi_k(x,t)| \lesssim_N 2^{kd}(1 + 2^k||x - |t||)^{-N} (1 + 2^k|t|)^{-d-1/2}.
\]
The proof will boil down to a number of applications of the non-stationary phase principle, which will reduce matters to estimating the contribution of the critical region of integration where the phase \( t|\xi| + x \cdot \xi \) is stationary.

i) As a warm-up, give a physical interpretation of \( \Phi_k \) when \( t \) is large. For the sake of the interpretation, replace the fast-decaying term \( (1 + 2^k||x - |t||)^{-N} \) by the characteristic function \( 1_{[-1,1]}(2^k||x - |t||) \) and draw its support in a space-time diagram. Show that the estimate is compatible with the energy conservation law for waves.
ii) Show by rescaling that it suffices to prove $\Box$ for the case $k = 0$.

iii) Show that the symmetries of the problem allow one to assume that $t > 0$ and $x = (x_1, 0, \ldots, 0)$ with $x_1 \geq 0$.

iv) Now you have an oscillatory integral with phase $\phi(\xi) = t|\xi| + x_1 \xi_1$. Show by using the non-stationary phase principle in $\xi_1$ that when $t \leq 1$ one has $|\Phi_0(x_1, t)| \lesssim_N (1 + x_1)^{-N}$ for any $N > 0$. This proves $\Box$ in the small time regime. Notice that the phase is not of the form $\lambda \phi$ and therefore one cannot use the non-stationary phase principle (Proposition 2.1) out of the box; however, the proof adapts easily.

v) Assume then that $t \geq 1$. We concentrate on proving the localization in $|x| - |t|$. Show, again by non-stationary phase principle in $\xi_1$, that if $t > 4x_1$ or $\frac{1}{4}x_1 > t$ then one has $|\Phi_0(x_1, t)| \lesssim_N (1 + |x_1 - t|)^{-N}$ for any $N > 0$. Show that this proves $\Box$ in this regime.

vi) Assume therefore that $t \geq 1$ and that $4x_1 > t > \frac{1}{4}x_1$. Show that the phase $\phi(\xi) = t|\xi| + x_1 \xi_1$ has critical points only if $t = x_1$ and the set of such critical points is contained in $(-\infty, 0] \xi_1$. We expect that the main contribution to the integral defining $\Phi_0$ comes from these critical points; therefore, smoothly excise a thin cone with axis $\mathbb{R} \xi_1$, say $\Gamma := \{ \xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1} \text{ s.t. } \xi_1 > (99/100)|\xi'| \}$, from the support of $\psi$ and show by non-stationary phase principle in $\xi'$ that the remaining integral (outside the cone) is dominated by $O_N((1 + t)^{-N})$ (which is acceptable since it is dominated by $\Box$ in this regime).

vii) Show by non-stationary phase principle in $\xi_1$ that one can do away with the portion of the cone $\Gamma$ where $\xi_1 > 0$. Indeed, in that region $|\nabla \phi| \gtrsim x_1$; show that this contribution is bounded by $O_N((1 + x_1)^{-N})$, which is again acceptable in this regime.

![Diagram]

viii) Finally, we come to the critical region (the portion of the cone where $|\xi| \sim 1$ and $\xi_1 < 0$). You have shown that if $x_1 = t$ and $\xi = (-\xi_1, 0)$ then $\nabla \phi = 0$. Now show that, even when $x_1 \neq t$, if $|\xi| \lesssim t^{-1/2}$ then $|\phi(-\xi_1, \xi') - \phi(-\xi_1, 0)| \leq 1/10$ (this means that the phase is barely changing). Taking advantage of this information, split (smoothly) the critical region dyadically into a sub-region where $|\xi'| \lesssim t^{-1/2}$ and sub-regions where $|\xi'| \sim 2t^{-1/2}$, where $j \in \mathbb{N}$ (and $2t^{-1/2} \lesssim 1$ because $|\xi'| \lesssim 1$ on the critical region). For the former, show that $\partial_{\xi_1} \phi(\xi) = x_1 - t + O(1)$. This is good when $|x_1 - t| \gtrsim t$, but if $x_1 - t$ is small then the integration by parts argument behind the non-stationary phase principle accumulates some unfortunate powers of $t$ (coming from the fact that $|\partial^j_{\xi'} \phi| \sim t$ for $j > 2$). To compensate for this, observe that there is oscillation in the $\xi'$ directions as well, and thus you should apply a non-stationary phase argument both in $\xi_1$ and $\xi'$. Show therefore that this contribution is bounded exactly by $\Box$. [If it helps, consider first the totally critical case $x_1 = t$.]

ix) For the remaining sub-regions, show that for each $j$ the quantity $\partial_{\xi_1} \phi(\xi) - (x_1 - t)$ is positive and has size $\sim 2^j$ (recall $2^j \lesssim t$). Use once again the non-stationary phase principle in $\xi_1, \xi'$ to estimate the contribution of each region. Finally, sum up in $j$ and show that the result is controlled by $\Box$ again.
HINTS

Hints for Exercise 1 Take the estimate

\[ |J(t)| \lesssim (1 + |t|)^{-(d-1)/2} \]

(\checkmark)
for granted, and follow these steps:

(i) Split the function \( f \in L^p(\mathbb{R}^d) \) as \( f(x) = f_1(x) + f_2(x) := f(x)1_{B(0,1)}(x) + f(x)(1 - 1_{B(0,1)}(x)) \), where \( 1_{B(0,1)} \) denotes the characteristic function of the ball of radius 1 centered at 0. Argue that \( f_1 \) is in \( L^1(\mathbb{R}^d) \) and therefore the contribution to \( f \) given by \( f_1 \) is continuous at \( \xi \neq 0 \).

(ii) Let \( \rho = |\xi| \) and write the radial Fourier transform \( g(\rho) \) of \( f_2 \) according to equation (1.1); then in the integral use a change of variable and argue that it will suffice to show the continuity of the function \( \Phi(\rho) := \rho^2 g(\rho) \).

(iii) It suffices to show continuity of \( \Phi \) from above and from below, so take \( \rho' > \rho \) and show that you can write (abusing notation a little, we write \( f_2 \) for the radial part of \( f_2(x) \) as well)

\[
\Phi(\rho) - \Phi(\rho') = \int_\rho^{\rho'} f_2 \left( \frac{s}{\rho} \right) s^{d-1} J(s) ds + \int_0^{+\infty} \left( f_2 \left( \frac{s}{\rho} \right) - f_2 \left( \frac{s}{\rho'} \right) \right) s^{d-1} J(s) ds.
\]

(iv) Fixing \( \rho \), use Hölder’s inequality and (\checkmark) (and the hypothesis on \( p \)) to show that the integral \( \int_\rho^{+\infty} |f_2 \left( \frac{s}{\rho} \right)| s^{d-1} J(s) ds \) is finite; hence deduce by standard arguments that term (3.2) tends to 0 as \( \rho' \to \rho^+ \).

(v) Using again Hölder’s inequality and (\checkmark), argue that term \( (3.3) \), also tends to 0 as \( \rho' \to \rho^+ \) (you might want to do a change of variables \( s = \rho t \) and then argue by the continuity the \( L^p \) norm with respect to dilations). Putting everything together, the proof is concluded.

Hints for Exercise 2 Prove upper- and lower-bounds for \( \sum_{n=1}^N r_2(n) \); tile the plane into squares with centers in the \( \mathbb{Z}^2 \) lattice.

Hints for Exercise 3 Try with \( D = a(x) \frac{d}{dx} \).

Hints for Exercise 4 Let \( a = 0, b = 1 \) for simplicity. Construct phases \( \phi_\lambda \) with oscillatory first derivatives where the oscillations happen at scale \( \lambda^{-1} \). A good choice would be \( \phi_\lambda(t) = 2\pi t + \lambda^{-1}\theta(\lambda t) \) with \( \theta \) a smooth periodic function of period 1. Show that

\[ \lambda I(\lambda) = \lambda \int_0^1 e^{i(2\pi t + \theta(t))} dt + O(1), \]

and furthermore show that you can choose \( \theta \) such that the integral above is \( \neq 0 \) and such that \( |\phi_\lambda'| > 1 \).

Hints for Exercise 5 Just adapt the proof of Proposition 2.2.

Hints for Exercise 6 Do a change of variable \( t = \theta s \) in \( I(\lambda) \), with \( \theta \) a parameter. If the inequality is to be invariant with respect to this transformation, a certain condition involving \( \alpha \) and \( k \) will have to be met.

Hints for Exercise 7 Let \( J_d \) be the function

\[ J_d(t) := \int_0^\pi e^{-2\pi i t \cos \theta} (\sin \theta)^{d-2} d\theta \]

and show by integration by parts that we have the recurrence relation

\[ (2\pi i t)^2 J_d(t) = (d - 3)(d - 5)J_{d-4}(t) - (d - 3)(d - 4)J_{d-2}(t); \]

iterate this to reduce to the case \( d = 3, 2 \). For \( J_3 \), compute the integral directly and get a bound of \( O(|t|^{-1}) \); for \( J_2 \), show by Van der Corput’s lemma that we have a
bound of $O(|t|^{-1/2})$ (besides the trivial bound of $O(1)$). Notice that you will need to split the integral in 3 parts (say, $\int_0^{\pi/4} + \int_{3\pi/4}^{\pi/4} + \int_{3\pi/4}^{\pi}$) in order to apply Van der Corput, since there is not a uniform lowerbound on $\phi'$ or $\phi''$ over the entire interval $[0, \pi]$.

Hints for Exercise 8

Find a rescaling of the $\xi$ variable that allows you to rewrite the phase as $|x|^3/2 \phi(\xi) = |x|^3/2(\text{sgn}(x)\xi + \xi^3)$, which is of the form we studied.

(ii) When $x > 1$, $\phi' \neq 0$. Show that this implies $\int |\exp(i|x|^{3/2} \phi(\xi))\varphi(\xi)| d\xi| \lesssim_N \min\{1, |x|^{-3N/2}\}$ by non-stationary phase. For the $\psi_j(\xi)$ terms instead, apply the proof of the non-stationary phase principle to each term, with $D$ the differential operator such that $D(e^{i\phi(\xi)}) = e^{i\phi(\xi)}$, to show that it contributes

$$
|x|^{-3N/2} \int |(D^k N \psi_j(\xi))| d\xi.
$$

To show that this is summable in $j$, show that one will have

$$(D^k N \psi_j(\xi)) = \sum_{k=0}^{N} c_{\phi,k,N}(\xi) 2^{-jk} \psi^{(k)}(2^{-j} \xi)$$

for some (integrable) functions $c_{\phi,k,N}$ depending on $\phi'$ that one can compute explicitly, and conclude using the fact that $\sum_{k \in \mathbb{N}} |\psi^{(k)}(2^{-j} \xi)| \lesssim_1 1$.

If you cannot convince yourself that $(D^k N \psi_j$ can be put in that form, try this other way: show by induction in $N$ that every term that appears in the expansion of $(D^k N \psi_j$ is of the form $(\psi_j)(\xi) \kappa^l(3\kappa^2 + 1)^{-m}$ (up to some constant) and that moreover $k - \ell + 2m > 1$. Then show that $\int |(\psi_j)(\xi)||\xi^l(3\kappa^2 + 1)^{-m}| d\xi \sim 2^{2(k-\ell+2m)}$, which is summable in $j$.

(iii) When $-1 < x < 1$, $\phi$ might have critical points (if $x < 0$), but only in the support of $\varphi$. Bound the $\varphi$ contribution by $O(1)$ trivially, and the contribution of the $\psi_j$ by the same argument above (since the non-stationary phase principle applies to them again).

(iv) When $x < -1$, $\phi$ certainly has two critical points ($\pm 1/\sqrt{3}$) inside the support of $\varphi$. For the $\psi_j$ terms, argue as above that they contribute $O_N(|x|^{-3N/2})$ and can thus be ignored. For $\varphi$, isolate the critical points by splitting $\varphi(\xi) = \varphi(100\xi) + (\varphi(\xi) - \varphi(100\xi))$ and similarly split the integral. Notice the support of $\varphi(100\xi)$ in $[-1/50, 1/50]$ and thus does not contain any critical points. For the contribution $\int \exp(i|x|^{3/2} \phi(\xi))(\varphi(\xi) - \varphi(100\xi)) d\xi$, notice that $|\phi''| \gtrsim 1$ and use Corollary 2 with $k = 2$.

Hints for Exercise 9

Here one needs to be a little clever: the point is to notice that the phase $2\pi r \cos \theta$ has first derivative non-zero around the problematic point $\theta = \pi/2$. If $\varphi$ denotes a smooth bump function supported in $[-\pi/4, \pi/4]$, split $J(r)$ as $\tilde{J}_1(r) + \hat{J}_2(r)$ where

$$
\tilde{J}_1(r) := \int e^{-2\pi i r \cos \theta} \varphi(\theta - \pi/2) d\theta.
$$

Argue by the non-stationary phase principle that $|\tilde{J}_1(r)| \lesssim_N (1 + r)^{-N}$ and that therefore $\int_0^R r \tilde{J}_1(r) d\theta = O(1)$. For $\hat{J}_2$, use Fubini and integration by parts to show that

$$
\int_0^R r \hat{J}_2(r) d\theta = -\int_0^\pi \left( \frac{R e^{-2\pi i R \cos \theta}}{2\pi i \cos \theta} + e^{-2\pi i R \cos \theta} \frac{1}{(2\pi i \cos \theta)^2} \right) (1 - \varphi(\theta - \pi/2)) d\theta.
$$

Where $1 - \varphi(\theta - \pi/2)$ does not vanish, one has that the 2nd derivative of the phase $\cos \theta$ is bounded from below; conclude therefore using Corollary 2 with term by term.

Hints for Exercise 10

...
i) You need to rewrite the measure of the sublevel set as a nice oscillatory integral. Show that \(| \{ t \in (a, b) : |\phi(t)| < \lambda \} \| = \int_a^b 1(|\phi(t)|/\lambda < 1)\, dt \). Choose some smooth non-negative bump function \( \varphi \) such that \( 1(|\phi(t)|/\lambda < 1) \leq \varphi(\phi(t)/\lambda) \) for every \( t \) and consequently replace the latter in the integral. Apply Fourier inversion to the term \( \varphi(\phi(t)/\lambda) \), followed by Fubini, and apply Van Der Corput’s lemma to the resulting oscillatory integral.

ii) Splitting \((a, b)\) as suggested, \( \int \phi(t) \) takes care of \(| \int_{|\phi'| > \theta} \exp(i\lambda\phi(t))\, dt | \) by taking the absolute value inside and \(| \{ \phi' \geq \theta \} \) consists of a bounded number of intervals (how many? See Remark 3, for each of which one can apply Proposition 2.2 to the associated oscillatory integral. Finally, choose \( \theta \) in order to optimize the resulting bound (that is, make it as small as possible).

iii) The base case \( k = 1 \) is easy. For the general \( k \) case, use the same strategy as in ii): split \((a, b)\) into those points where \(|\phi'(t)| < \theta\), to which the inductive hypothesis applies, and those where \(|\phi'(t)| \geq \theta\), which consist of boundedly many intervals on each of which you can apply the base case.

iv) What does it mean for \( x \in \mathbb{Z}_p \) to have \(|x|_p \leq p^{-1}\)? And what does it mean for it to have \(|x|_p \leq p^{-2}\)? etc.

**Hints for Exercise 11**

i) Squeezing \( E \) into an interval can only make the left hand side of the inequality smaller, so it is fine to do. Once \( E \) is an interval, just take uniformly distributed points inside it as your \( x_j \)'s and perform the simple calculation that results.

ii) By Lagrange interpolation, you can find a polynomial \( P(X) \) of degree \( k \) such that \( P(x_j) = f(x_j) \) for all \( j \in \{0, \ldots, k\} \). Argue that \( P^{(k)}(X) = f^{(k)} \) must have a zero inside \((x_0, x_k)\). Then notice that \( P^{(k)}(X) \) must be a constant, precisely \( k! \) times the coefficient of \( X^k \). Calculate that coefficient explicitly (the constraints \( P(x_j) = f(x_j) \) give you a linear system of equations in the coefficients of \( P \), with a nice Vandermonde matrix appearing; solving the system with Cramer’s rule is a breeze).

iii) Part i) gives you the points \( x_j \) and for each \( j \) reversing the inequality one has \( \left( \prod_{i \neq j} |x_i - x_j| \right)^{-1} \leq (2e^k)|E|^{-k} \). Then use ii) together with this inequality, the fact that \(|\phi^{(k)}| > 1 \) and that \(|\phi(x_j)| < \lambda \) for each \( j \).

v) Remember Remark 2 how many intervals is \(||\phi'| \geq \theta \) made of at most?

vi) Use \((a, b) = (0, 1)\) and \( \phi(t) = e^{it} \). To do this you will need complex integration.

Use a regularisation: evaluate instead \( \int_0^1 e^{it} e^{-\alpha t^2} \, dt \) and take \( \epsilon \to 0 \) at the end. To evaluate the integral, see it as the integral of holomorphic function \( e^{z^2} \) along the path \( \Gamma_0 := \{ (i - \epsilon)^{1/2}t \text{ s.t. } t \in [0, 1] \} \) (fixing a branch of the \( k \)-th root). Complete the path \( \Gamma_0 \) to a well-chosen closed path \( \Gamma \) that includes part of the real line and conclude using Cauchy’s integral theorem and some simple estimates on the decay of \(|e^{z^2}| \) along certain directions of the complex plane.

**Hints for Exercise 12**
The first step is as in the proof of Proposition 2.2 and one has
\[
\int_a^b e^{i\lambda\phi(t)}\psi(t)\, dt = \left( \frac{e^{i\lambda\phi(b)}\psi(b)}{i\lambda\phi'(b)} - \frac{e^{i\lambda\phi(a)}\psi(a)}{i\lambda\phi'(a)} \right) - \int_a^b e^{i\lambda\phi(t)} \left( \frac{\psi}{i\lambda\phi'} \right)'(t)\, dt.
\]
Iterating this on the integral on the right hand side shows that it is of size \( O(\lambda^{-2}) \) and therefore the above gives you \( a_0, b_0 \). Keep applying integration by parts until necessary.

**Hints for Exercise 13**
i) You should see the integral as the complex integral of (a multiple of) the holomorphic function \( z^a e^{-z^2} \) along the path \( \Gamma_0 := (1-i\lambda)^{1/2} \mathbb{R} \subset \mathbb{C} \). Truncate
to $\Gamma^+_0 := \{(1-i\lambda)^{1/2}t : t \in [0,R]\}$ and complete this path to a closed path $\Gamma$ that includes the real interval $[0,R]$. Use Cauchy’s integral theorem and simple estimates on $|z^m e^{-z^2}|$ to conclude that as $R \to \infty$ this (plus an equal contribution from a similarly defined $\Gamma^{-}_0$) gives precisely what is claimed.

The power series expansion of $(1-i\lambda)^{-(m+1)/2}$ is standard, just write it as $\lambda^{-(m+1)/2}(\lambda^{-1} - i)^{-(m+1)/2}$ and expand the function $(z - i)^{-(m+1)/2}$ in the $z$ variable.

ii) It is straightforward to show that $\int |t|^m \eta(t)|\varphi(t/\delta)dt \lesssim_{m,\eta} \delta^{m+1}$. The other term looks a bit nasty, but luckily one does not have to make all the computations implied. Let $\eta(t)(1-\varphi(t/\delta)) =: \omega_\delta(t)$ for convenience. Show that the support of $\omega_\delta(t)$ and all its derivatives is contained in $\delta \lesssim |t| \lesssim 1$. The integration by parts argument in the proof of Proposition 2.1 reduces matters to estimating $\lambda^{-N} \int |(D^k)^N(t^m \omega_\delta(t))|dt$ with $Df(t) := (2it)^j f'(t)$. $(D^k)^N$ is too horrible to evaluate precisely (this is due to the fact that $D^k$ is not a derivation, that is it does not satisfy Leibniz’s rule; see Exercise 3), but we do not need to. Show by induction on $N$ that every term in the expansion of $(D^k)^Nf$ is of the form $t^{(j)}t^{-(2N-j)}$ for $j \in \{0,\ldots,N\}$. Apply this to function $t^m \omega_\delta(t)$ together with the general Leibniz’s rule (the binomial expansion of $(fg)^{(j)}$) to conclude that $(D^k)^N(t^m \omega_\delta(t))$ is of the form

$$
\sum_{k=0}^m \sum_{j=0}^N c_{j,k,N} \omega_\delta^{(j-k)}(t)t^{-(2N-m)+(j-k)}
$$

for some constants $c_{j,k,N}$ whose precise value does not concern us. Show that, by the support of $\omega_\delta^{(j-k)}$ and the fact that $2N > m + 1$, the integral of each term is at most $\lambda^{-N}\delta^{-(2N-m)+1}$ (careful with $\omega_\delta^{(j-k)}$, when expanded this contains factors of $\delta^{-\ell}$ for $\ell \leq j-k$). Therefore the quantity to be estimated is controlled by $\delta^{m+1} + \lambda^{-N}\delta^{-(2N-m)+1}$, and it remains for you to choose a good $\delta$ that minimises this.

iii) Just repeat the second part of the above with $\delta \sim 1$.

iv) Taylor expansion gives you $e^{-t^2} \psi(t) = P(t) + t^{n+1}R_n(t)$ for some polynomial $P$ of degree $n$ and a remainder $R_n(t)$ that is well-behaved. Write the term $\int \exp(i\lambda t^2)\exp(-t^2)(P(t)\eta(t))dt$ as

$$
- \int e^{i\lambda t^2} e^{-t^2} P(t)dt - \int e^{i\lambda t^2} e^{-t^2} P(t)(1-\eta(t))dt;
$$

use i) on each monomial of $P(t)$ to deal with the first integral, and use iii) on the second integral to show it is an error term (show that $\exp(-t^2)P(t)(1-\eta(t))$ is a Schwartz function).

For $\int \exp(i\lambda t^2)t^{n+1}\exp(-t^2)R_n(t)\eta(t))dt$, use ii) to show that it is also an error term. Choosing $n$ large enough gives the result.

v) Simply show that under the given hypotheses $s = |\psi(t)|^{1/2}$ defines a diffeomorphism. Finally, if $t = \theta(s)$ denotes the inverse diffeomorphism, show that $\psi(\theta(s))\theta'(s)$ satisfies the same properties as $\psi$. Performing the change of variable $t \mapsto s$ concludes the argument.

vi) Repeat the proof you gave for i) basically word by word and adapt the rest.

**Hints for Exercise 14**

i) The important thing is that $dt/t$ is invariant with respect to changes of variables of the form $t \mapsto \alpha t$.

ii) Since $P$ is a polynomial, we have $P^{(d)}(t) = d!$. With $\psi(t) = 1/t$, the constant in Corollary 2.4 is $\sim_d 1/R = O_d(1)$.

iii) $e^{dP(t)} - e^{dP_{d-1}(t)} = e^{dP_{d-1}(t)}(e^{dt} - 1)$ and $\int_0^1 |t|^{d} dt = O_d(1)$.
viii) You can consider

b) Summing over

d) The bound on the length of

vii) We can add or subtract

point

ω

of pieces (see

⟨

a) The bound on the length of

where

ψ

is a smooth bump function supported around the origin and

vi) The remaining terms are bounded by


c) The first sum runs over those

i) to evaluate

summation and observe that the resulting expression is a telescopic sum, thus

correct sign) and viceversa.

that appears in this sum appears exactly once in the right hand side (with the

e) It must be the case that both

C|J|λ1/2 < |I| and

λ−1/2 < |I|...

HINTS FOR EXERCISE 15

i) Just a boring high-school trigonometry exercise.

ii) Write

\sum_n e(f(n)) = \sum_n g(n)(e(f(n)) - e(f(n-1)))

and check that every term that appears in this sum appears exactly once in the right hand side (with the correct sign) and viceversa.

iv) Write

f(n) - f(n-1) = \int_n^\infty f'(s)ds.

v) Apply triangle inequality to the left hand side, thus erasing the e(f(n)) factors. Then appeal to monotonicity to take the absolute values outside of the summation and observe that the resulting expression is a telescopic sum, thus only the first and last terms survive. Finally, use the trigonometric identity in

i) to evaluate

g(N), g(0), combine with periodicity of cot and iv).

vi) The remaining terms are bounded by |g(N)| + |g(1)| and are dealt with as in the final part of v).

a) The bound on the length of J is an immediate consequence of the mean value theorem.

b) \( A_k \cup B_k \) has length 1 and they are all disjoint.

c) The first sum runs over those n such that |f'(n) - k| < \( \delta \). Since |f''| > \( \lambda \), there are at most \( O(\delta/\lambda) \) such values of n.

For the second sum, verify that the hypotheses of \( \sqrt{1/|J|} \) as in vi) apply.

d) Summing over k gives the factor of \( C\lambda|J| + 1 \) by part b).

e) It must be the case that both \( C|J|\lambda^{1/2} < |I| \) and \( \lambda^{-1/2} < |I| \).

HINTS FOR EXERCISE 16

Try with inner product \( (P,Q) := |P(\nabla)||Q| \), where if

\[ P(X) = \sum_{\alpha:|\alpha|=k} a_\alpha X^\alpha \]

then \( P(\nabla) = \sum_{\alpha:|\alpha|=k} a_\alpha \partial^\alpha \). What does it mean that

\[ \langle (\xi \cdot X)^k, Q \rangle = 0 \]?

HINTS FOR EXERCISE 17

Decompose the sphere smoothly in a bounded number of pieces (see Exercise 20 part ii) for an idea how). Do a change of variables: a point \( \omega \in S^{d-1} \) is given by \( \omega = (\bar{x},\phi(\bar{x})) \) where \( \bar{x} \in \mathbb{R}^{d-1} \) and \( \phi(\bar{x}) = (1-x_1^2-\cdots-x_{d-1}^2)^{1/2} \). Performing the change of variables, one has that

\[ \int_{S^{d-1}} \exp(-2\pi i\ell \phi(\bar{x})) d\sigma(\omega) \]

is a sum of integrals of the form

\[ \int_{\mathbb{R}^{d-1}} e^{-2\pi i\ell \phi(\bar{x})} S(\bar{x}) \psi(\bar{x}) d\bar{x}, \]

where \( \psi \) is a smooth bump function supported around the origin and \( S(\bar{x})d\bar{x} \) gives the surface measure of \( S^{d-1} \) in the \( \bar{x} \)-coordinates. Justify all of the above, check the Hessian of \( \phi \) and apply Theorems 5,3 directly.

HINTS FOR EXERCISE 18

ii) just solve \( |u(x,y)| < \lambda \) for \( x,y \).

iii) Show that for \( x \) close to 1 the function \( u(x,y) \) is constant in \( x \) and periodic in \( y \) with period \( 1/N \). Thus concentrate on a single tooth and show that for
these x the set of y in this tooth such that |u(x, y)| < \lambda has measure bounded from below by \lambda.

**Hints for Exercise 19**

ii) Just perform the integration \int_{R^d} (1 + \lambda|z|)^{-N} dz with N large. If you prefer, you can split R^d into dyadic annuli \{z \in R^d : 2^j < |z| \leq 2^{j+1}\}, estimate the integral over each annulus and sum in j \in Z.

iii) \nabla y e^{i\lambda(u(y)z - u(x))} = i\lambda e^{i\lambda(u(y)z - u(x))} \nabla y (u(y)z - u(x)), so take the inner product of this vector by vector (i\lambda)^{-1} [\nabla y (u(y)z - u(x))] to get the pure exponential factor back.

iv) From the previous point, \theta_x(y) is the vector \nabla y (u(y)z - u(x)). For the numerator, observe that by hypothesis we can control |\partial_{\theta_x} u| \lesssim \alpha, w, \psi 1 for any multi-index \beta; show by the Fundamental Theorem of Calculus that \partial_{\theta_x}^\alpha u(y + z) = \int_0^1 \nabla_y (\partial_\theta^\alpha u(y + tz)) \cdot zdtd and deduce that |\partial_{\theta_x}^\alpha u(y + z) - u(y)| \lesssim \alpha, w, \psi |z|.

v) Show that D^\gamma f = (i\lambda)^{-1} \nabla \cdot (f \theta_x) \psi (y + z) \psi (y) for simplicity. Show by induction that (D^\gamma)^N \Psi contains only terms of the form \partial_{\theta_x}^\alpha \Psi \cdot \partial_{\theta_x}^\beta \theta_x \cdot \ldots \cdot \partial_{\theta_x}^\gamma \theta_x with \alpha_0 + \alpha_1 + \ldots + \alpha_N = N, and deduce by iv) that |(D^\gamma)^N \Psi| is thus controlled by a sum of terms of the form |z|^{-N} |\partial_{\theta_x}^\alpha \Psi|.

**Hints for Exercise 20**

i) The integral factorises into a product of integrals that are exactly like the ones in i) of Exercise 13.

ii) The decomposition of \nabla \beta z \cones follows from a trivial pigeonholing argument. Building the partition of unity \sum_{k=1}^d G_k(x) is also a standard exercise: simply take non-negative bump functions \rho_k on the sphere \Omega^d with support contained in \Gamma_k \cap \Omega^d and identically 1 on \Gamma_k, then define \Gamma_k := \rho_k(x/\|x\|)\sum_{j=1}^d p_j(x/\|x\|)^{-1} and show this is C^\infty and satisfies the other conditions.

iii) See the hints for ii) of Exercise 13 for this is extremely similar. The trivial estimate near the origin will give you a contribution \int |x| \alpha \phi(x/\delta) dx \lesssim \delta^{d+\alpha}.

iv) Again, repeat the second part of the proof you gave for step iii) above, but with \delta \sim 1.

v) See the hints for part iv) of Exercise 13 and follow them closely.

vi) Morse’s lemma says precisely what we want: if f \in C^\infty (R^d) is such that f(0) = 0, \nabla f(0) = 0 but det(Hess(f)) \neq 0 (where Hess(f) denotes the Hessian of f), then there exists a neighbourhood U of 0 and a C^\infty diffeomorphism \Phi : R^d \rightarrow R^d such that \Phi(0) = 0 and f \circ \Phi^{-1}(y) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_d^2 for all y \in \Phi(U). Moreover, (k, d - k) is the signature of Hess(f).
Use the change of variable provided by the diffeomorphism Φ to turn the phase ω(x) into a quadratic phase and conclude by appealing to the previous steps.

Hints for Exercise 21

i) At time t the wave solution is concentrated in a shell of width ∼ 1 and radius ∼ t, and it has amplitude ∼ t^{-(d-1)/2}. Multiply the square of the amplitude by the volume occupied by the wave to get an estimate for the energy.

ii) The gradient of the phase is \( \nabla \phi(\xi) = t \frac{1}{\xi_1} + x_1 e_1 \), so if \( t \lesssim 1 \) and \( x_1 \gtrsim 1 \) one has \( \partial_{\xi_1} \phi \) is bounded from below by \( \gtrsim x_1 \). Show that for higher derivatives one has \( |\partial_{\xi_1}^k \phi| \lesssim 1 \). Apply the integration by parts argument of Proposition 21.1 to the integral in \( \xi_1 \) with respect to the differential operator \( D_1 = (i \partial_{\xi_1} \phi(\xi))^2 \partial_{\xi_1} \). Show by induction in \( \ell \) that \( (D_1^\ell)^N \psi \lesssim_{N, \psi} \frac{1}{t} \) by now you should know how to proceed: show that \( (D_1^\ell)^N \psi \) consists of terms of the form \( (\partial_{\xi_1} \phi)^{-\ell} \partial_{\xi_1}^j \psi \prod_{k=2}^m (\partial_{\xi_1}^k \phi)^{\beta_k} \) where \( \ell \geq N \). Finish by integrating in the remaining variables \( \xi_2, \ldots, \xi_d \).

v) Show that one always has \( |\partial_{\xi_j} \phi(\xi)| \gtrsim |\xi_1 - t| \gtrsim t \) in this regime. Then repeat the analysis done in iv), but be more careful this time: now you can only say (show this) that \( |\partial_{\xi_1} \phi| \lesssim t \). In order not to lose the decay, you will have to show by induction that the terms of the form \( (\partial_{\xi_j} \phi)^{-\ell} \partial_{\xi_1}^j \psi \prod_{k=2}^m (\partial_{\xi_1}^k \phi)^{\beta_k} \) appearing in the expansion of \( (D_1^\ell)^N \psi \) actually satisfy \( \ell - \sum_{k=1}^m k \beta_k = 0 \) and \( \beta_1 + \sum_{k=2}^m (k-1) \beta_k = N \), thus yielding the correct decay of \( |x_1 - t|^{-N} \).

iv) For the smooth excision of a cone, refer to the hints for parts ii)-iii) of Exercise 20. For the estimate, observe that ignoring the \( \xi_1 \) component one has \( |\nabla \phi(\xi)| \gtrsim \frac{t}{|\xi|} |\xi_1| \) and outside the cone one has \( |\xi_1| \gtrsim |\xi| \). Upon a further smooth partition of unity into cones with axes the coordinate directions \( \xi_2, \ldots, \xi_d \), one can reduce to the case where \( |\xi_1| \gtrsim |\xi| \) and thus \( |\partial_{\xi_1} \phi(\xi)| \gtrsim t \). Then apply the same argument as in v) above with differential operator \( D_j = (i \partial_{\xi_j} \phi(\xi))^2 \partial_{\xi_j} \) to conclude.

vii) Same reasoning as in vi).

viii) You are allowed to use Taylor expansion of \( |\xi| = \xi_1 (1 + (|\xi|^2/\xi_1^2))^{1/2} \) in terms of \( |\xi|^2/\xi_1^2 \), both for estimating the difference in the phases and for estimating \( \partial_{\xi_1} \phi \). The non-stationary phase principle in both \( \xi_1, \xi_j \) (after once again using a smooth partition of unity to reduce to the case where \( |\xi_j| \gtrsim |\xi| \)) will result in having to estimate (for example) \( \int |(D_1^2 D_1^j)^N \psi(\xi)| \, d\xi \). This is quite excruciating, but you can again use an inductive argument to show that the integrand consists entirely of terms of the form \( (\partial_{\xi_j} \phi)^{-\ell} (\partial_{\xi_1} \phi)^{-\ell'} \cdot \partial_{\xi_1}^j \partial_{\xi_1}^{j'} \psi \prod_{k=2}^m (\partial_{\xi_1}^k \phi)^{\beta_k} \) where the various parameters involved satisfy certain favourable constraints. Then one can see that \( |\partial_{\xi_1} \phi| \gtrsim |x_1 - t|, \, |\partial_{\xi_1}^2 \phi| \lesssim 1 \) (this because of the restriction to \( |\xi_1| \lesssim t^{-1/2} \)), \( |\partial_{\xi_j} \phi| \sim t \) and in general \( |\partial_{\xi_1}^k \partial_{\xi_1}^{k'} \phi| \lesssim t \) for all \( k + k' \geq 2 \). Using these facts and the induction one can verify that \( |(D_1^2 D_1^j)^N \psi(\xi)| \lesssim |x_1 - t|^{-N} \). The factor of \( t^{-(d-1)/2} \) will come from the fact that the region of integration is approximately a cylinder of length 1 and radius \( t^{-1/2} \).

ix) This part is a repetition of part viii) above essentially, except for the fact that now we can assume \( \xi_j \sim 2t^{-1/2} \) instead and consequently one has to update the upper- and lower-bounds on the derivatives of the phase accordingly.

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