Double Poisson gebras up to homotopy are pre-Calabi–Yau algebras

Johan Leray

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Contents

0 – Introduction

What is noncommutative (derived) Geometry ?

To an associative algebra A, one can associate a family of schemes called *representation schemes*

$$\mathrm{Rep}_n(A) : C \mapsto \mathrm{Hom}_{\mathsf{AssAlg}_{\mathbf{k}}}(A, \mathscr{M}_n(k) \otimes_k C).$$

The Kontsevich–Rosenberg principle says that a noncommutative Poisson structure on A is a structure such that the affine scheme $\mathrm{Rep}_n(A)$ is Poisson.

(Shifted) Poisson Geometry

In algebraic (commutative) geometry, there is two ways to define a Poisson structure :

Polyvector field

Poisson structure is a bivector field satisfying a Maurer–Cartan equation

Bracket

Poisson structure on a commutative algebra is a bracket $\{-,-\} : A^{\otimes 2} \to A$ which satisfies some relations.

encoded by the operad Pois

In derived (commutative) geometry :

Shifted Polyvector field

Shifted Poisson structure is a shifted bivector field satisfying a Maurer–Cartan equation [Calaque et al.]

Brackets

Homotopy Poisson structure is encoded by the operad Pois_∞, resolution of Pois.

These two definitions coincide. [Melani]

Noncommutative Poisson structure

In 2006, Van den Bergh define the noncommutative Poisson structure, called *double Poisson structure* :

NC Polyvector field

NC Poisson structure is a nc bivector field satisfying a Maurer–Cartan equation.

Double bracket

Double Poisson structure on a associative algebra is a double bracket $\{\!\{-,-\}\!\} : A^{\otimes 2} \to A^{\otimes 2}$ which satisfies some relations.

encoded by a properad

These two structures, which induce Poisson structure on $\mathrm{Rep_n(A)}$, coincide if the underlying associative algebra is smooth.

Derived Noncommutative Poisson structure

There is two ways to define what is a derived noncommutative Poisson structure :

"Shifted NC Polyvector field"
Generalisation of nc bivector field satisfying a Maurer–Cartan equation
$\longrightarrow$ **pre-Calabi–Yau algebra**
(Kontsevich–Vlassopoulos)

Double brackets
Homotopy double Poisson structure
$\longrightarrow \mathrm{DPois}_\infty$**-gebras**
(L.)

Do these two structures coincide?

Yeung (and also Pridham) shown that a pre-Calabi–Yau structure on an associative algebra induces a shifted Poisson structure on its derived representation scheme (defined by Berest *et al.*).

Goal of this talk

The goal of this talk is to explain the following theorem

Theorem [L.–Vallette]

pre-Calabi–Yau algebras = curved homotopy double Poisson gebras.

This theorem follows some results in this direction :

- Iyudu–Kontsevich–Vlassopoulos shown that double Poisson gebras are pre-Calabi–Yau algebras.
- Fernández–Herscovich shown that infinity double Poisson gebras (defined by Schedler) and quasi-double Poisson gebras (defined by Van den Bergh) are pre-Calabi–Yau algebras.

This properadic description of pre-Calabi–Yau algebras has also some important consequences.

1 – Double Poisson gebra up to homotopy

Double Poisson gebra

Definition (Double Poisson gebra)

A *double Poisson gebra* amounts to a data $(A, \mu, \{\!\{-,-\}\!\})$ made up of a dg associative algebra (A, μ) and a morphism called the *double bracket*

$$\{\!\{-,-\}\!\}: A \otimes A \longrightarrow A \otimes A \ ,$$

satisfying, for any $a, b, c \in A$

$\{\!\{a, b\}\!\} = \pm \{\!\{b, a\}\!\}'' \otimes \{\!\{b, a\}\!\}'$

$\{\!\{a, \mu(b, c)\}\!\} = \pm \mu(b, \{\!\{a, c\}\!\}') \otimes \{\!\{a, c\}\!\}'' + \{\!\{a, b\}\!\}' \otimes \mu(\{\!\{a, b\}\!\}'', c)$

and a relation called *double Jacobi*.

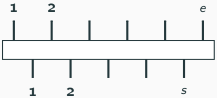
This structure is encoded by a *properad*.

What is a properad ?

Definition (Properad [Vallette])

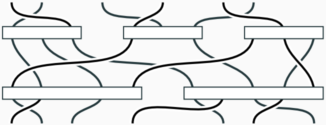
A *properad* is an algebra over the monad $\mathscr{G}$ of connected directed graphs, which is equivalent to a monoid in the category of $\mathfrak{S}$-bimodules with the monoidal product $\boxtimes$.

A properad $P = \{P(s, e)\}_{s,e \in \mathbb{N}^*}$ is a $\mathfrak{S}$-bimodule where an element of $P(s, e)$ can be represented by



$\in \quad P(5, 6)$

equipped with a product $P \boxtimes P \longrightarrow P$:

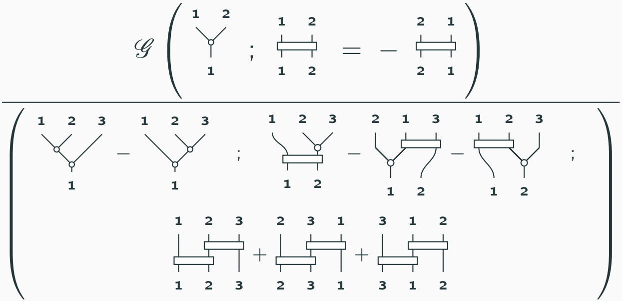


$\mapsto$

For A a dg vector space :

$$\mathrm{End}_A = \left\{\mathrm{Hom}_k(A^{\otimes e}, A^{\otimes s})\right\}_{s,e\in\mathbb{N}^*}$$

$\mathrm{DPois} =$



Proposition

A double Poisson structure on a dg vector space A corresponds to a morphism of properads $\mathrm{DPois} \to \mathrm{End}_A$.

Double Poisson gebra up to homotopy

A double Poisson structure up to homotopy is encoded by a cofibrant replacement of DPois.

Theorem [L.]

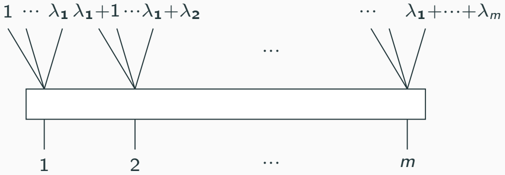
The properad DPois is Koszul. Then the *minimal* cofibrant replacement of DPois is

$$\mathrm{DPois}_\infty \longrightarrow \mathrm{DPois}.$$

Properad DPois_∞ : the generators

The dg properad $\mathrm{DPois}_\infty = \left(\mathscr{G}\left(\mathrm{s}^{-1}\overline{\mathrm{DPois}}^{\mathsf{i}}\right), \partial_\Delta\right)$ is the quasi-free properad constructed on the *dg coproperad* $\mathrm{DPois}^{\mathsf{i}}$:

- generated by

$\nu_{\lambda_1,\ldots,\lambda_m} =$  ,

with $\lambda_i \geqslant 1$ and with the cyclic symmetry

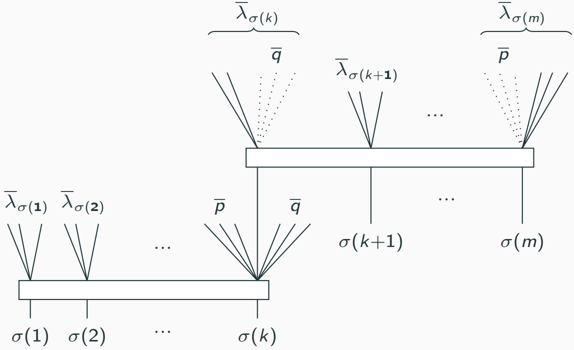
$$\nu_{\lambda_1,\ldots,\lambda_m} = \pm \nu_{\lambda_2,\ldots,\lambda_m,\lambda_1}.$$

Properad DPois_∞ : the differential

- the differential is constructed with the *partial coproduct* $\Delta_{(1,1)}$ of $\mathrm{DPois}^{\mathrm{i}}$.

$\Delta_{(1,1)}\left(\nu_{\lambda_1,\ldots,\lambda_m}\right) =$

$$\sum_{\substack{k=1 \\ \sigma\in\mathbb{Z}/\mathbb{Z}_m}}^{m} \sum_{\substack{0\leqslant p<\lambda_{\sigma(m)} \\ 0\leqslant q<\lambda_{\sigma(k)}}} \pm$$



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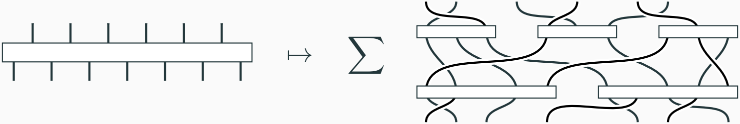
Remark about coproperad

The $\mathfrak{S}$-bimodule $\mathrm{DPois}^{\mathrm{i}}$ is a *coproperad*, i.e. a comonoid for the monoidal structure $\boxtimes$: $\mathrm{DPois}^{\mathrm{i}}$ is equipped with a coproduct

$$\Delta : \mathrm{DPois}^{\mathrm{i}} \longrightarrow \mathrm{DPois}^{\mathrm{i}} \boxtimes \mathrm{DPois}^{\mathrm{i}}$$



Remark

The coproduct of $\mathrm{DPois}^{\mathrm{i}}$ is difficult to describe ...

Two descriptions of DPois_∞ gebras

Definition (Double Poisson structure up to homotopy)

A double Poisson structure up to homotopy on a dg vector space A is a morphism of properads $\mathrm{DPois}_\infty \longrightarrow \mathrm{End}_A$.

Proposition – Rosetta Stone [Vallette]

$$\mathrm{Hom}_{dg\ \ properads}(\mathrm{DPois}_\infty, \mathrm{End}_A) \cong \mathrm{MC}\,(\mathfrak{DPois})$$

where $\mathfrak{DPois}$ is the Lie-admissible algebra

$$\mathfrak{DPois} = \left(\prod_{n,m \geqslant 1} \mathrm{Hom}_{\mathfrak{S}_m^{\mathbf{op}} \times \mathfrak{S}_n} \left(\overline{\mathrm{DPois}}^{\text{i}}(m,n), \mathrm{Hom}\left(A^{\otimes n}, A^{\otimes m}\right)\right), \partial, \star \right)$$

where $f \star g = \overline{\mathrm{DPois}}^{\text{i}} \xrightarrow{\Delta_{(\mathbf{1},\mathbf{1})}} (\overline{\mathrm{DPois}}^{\text{i}})^{\boxtimes 2} \xrightarrow{f \boxtimes g} (\mathrm{End}_A)^{\boxtimes 2} \xrightarrow{\mu} \mathrm{End}_A$

2 – Pre-Calabi–Yau algebra

First definition

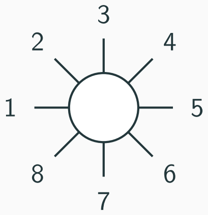
Definition ("Almost pre-Calabi–Yau algebra")

A structure of *almost pre-Calabi–Yau algebra* on a dg vector space A is a cyclic A_∞-structure on $sA \oplus A^*$ equipped with its canonical degree -1 skew-symmetric pairing such that A is a sub A_∞ algebra.

Definition (Cyclic non symmetric operad)

A *cyclic non-symmetric operad* is an algebra over the monad of planar trees.

Let $\mathscr{P}$ a cyclic ns operad, that is a collection of dg vector spaces $\mathscr{P}(\langle n \rangle)$ with an action of $\mathbb{Z}/n\mathbb{Z}$ where elements are represented by corollas ...

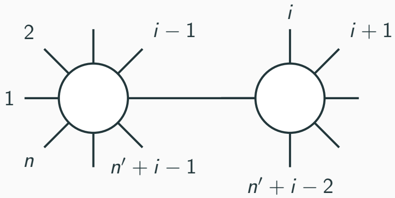


$\in \mathscr{P}(\langle 8 \rangle)$

Cyclic non symmetric operad

with several compositions maps

$$\circ_i \,:\, \mathscr{P}(\langle n\rangle)\otimes\mathscr{P}(\langle n'\rangle) \to \mathscr{P}(\langle n+n'-2\rangle)\,,\ \text{for}\ n \geqslant 2\,,\ n' \geqslant 1\,,\ \text{and}\ 2 \leqslant i \leqslant n\,.$$



Remark

A cyclic operad is not a monoid.

Two examples of cyclic operads

- Let $(V, d_V, \langle\ ,\ \rangle)$ be a differential graded vector space equipped with a symmetric bilinear form of degree 0 . Its *endomorphism cyclic non-symmetric operad* $\mathscr{End}_V$ is

$$\mathscr{End}_V(\langle n \rangle) = V^{\otimes n}$$

by the partial composition map

$$\circ_i (a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_{n'}) = \\ \pm \langle a_i, b_1 \rangle a_1 \otimes \cdots \otimes a_{i-1} \otimes b_2 \otimes \cdots \otimes b_{n'} \otimes a_{i+1} \otimes \cdots \otimes a_n \ .$$

- Cyclic associative : $\mathscr{As}(\langle n \rangle) = k\mu_n$ with trivial $\mathbb{Z}/n\mathbb{Z}$ action, for $n \geqslant 3$, and $\mathscr{As}(\langle 2 \rangle) = \mathscr{As}(\langle 1 \rangle) = 0$.

It forms a cyclic non-symmetric operad once equipped with the following partial composition maps

$$\mu_n \circ_i \mu_{n'} = \mu_{n+n'-2} \ .$$

Algebra over a cyclic ns operad

Definition (Algebra over a cyclic non-symmetric operad)

An *algebra structure over a cyclic non-symmetric operad* $\mathscr{P}$ on a differential graded vector space $(V, d_V, \langle\ ,\ \rangle)$ equipped with a symmetric bilinear form is given by the data of morphism of cyclic non-symmetric operads $\mathscr{P} \to \mathscr{E}nd_V$.

Example

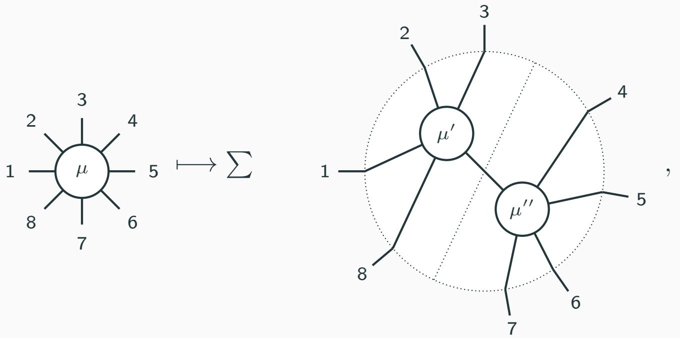
Algebras over the cyclic non-symmetric operad $\mathscr{A}s$ are *cyclic associative algebras*, which are differential graded associative algebras $(V, d_V, \cdot)$ equipped with a symmetric bilinear form $\langle\ ,\ \rangle$ such that

$$\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle \quad \forall a, b, c \in V.$$

Dual of $\mathscr{A}s$

To the cyclic operad $\mathscr{A}s$, we associate its (anti)-cyclic cooperad dual $\mathscr{A}s^{¡}$, i.e. the cyclic module $\mathscr{A}s^{¡}(\langle n \rangle) = ks^{n-2}$, with partial decompositions maps

$$\delta_i \;:\; \mathscr{A}s^{¡}(\langle n+n'-2 \rangle) \to \mathscr{A}s^{¡}(\langle n \rangle) \otimes \mathscr{A}s^{¡}(\langle n' \rangle)$$



satisfying some relations.

Cyclic A_∞ algebra

Let $(V, d_V, \langle -,-\rangle)$ be a dg vector space with a symmetric bilinear form.

A *cyclic A_∞ structure* on V is a Maurer–Cartan element of the following Lie algebra

$$\mathrm{MC}\left(\prod_{n\geqslant 1} \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}\left(\mathscr{As}^{\text{!`}}(\langle n\rangle), \mathscr{End}_V(\langle n\rangle)\right), \partial, \{-,-\}\right)$$

where

$$\{\mu,\nu\} = \sum_{i=2}^{n} \circ_i(\mu\otimes\nu)\delta_i - (-1)^{|\mu||\nu|}\sum_{j=2}^{n'} \circ_j(\nu\otimes\mu)\delta_j$$

Cyclic A_∞ algebra on $sA \oplus A^*$

Let $sA \oplus A^*$ equipped with its canonical degree -1 skew-symmetric pairing $\langle f, sx\rangle = (-1)^{|f|} f(x)$.

Proposition

The shifted Lie algebra controlling cyclic A_∞ structures is isomorphic to

$$\mathrm{s}^2 \prod_{\substack{N \geqslant 3 \\ N=n+m}} \left(\bigoplus_{1 \leqslant m < N} \left(\bigoplus_{\lambda_1 + \cdots + \lambda_m = n} A \otimes ((\mathrm{s}A)^*)^{\otimes \lambda_1} \otimes \cdots \otimes A \otimes ((\mathrm{s}A)^*)^{\otimes \lambda_m} \right)^{\mathbb{Z}/m\mathbb{Z}} \oplus \left(((\mathrm{s}A)^*)^{\otimes N} \right)^{\mathbb{Z}/N\mathbb{Z}} \right)$$

with the bracket given by $\left\{\mathrm{s}^2 a_1 \otimes \cdots \otimes a_N, \mathrm{s}^2 b_1 \otimes \cdots \otimes b_{N'}\right\} =$

$$\mathrm{s}^2 \sum_{i=2}^{N} \pm \langle a_i, b_1\rangle \, a_1 \otimes \cdots \otimes a_{i-1} \otimes b_2 \otimes \cdots \otimes b_{N'} \otimes a_{i+1} \otimes \cdots \otimes a_N$$

$$+ \, \mathrm{s}^2 \sum_{j=2}^{N'} \pm \langle b_j, a_1\rangle \, b_1 \otimes \cdots \otimes b_{j-1} \otimes a_2 \otimes \cdots \otimes a_N \otimes b_{j+1} \otimes \cdots \otimes b_{N'} \; ,$$

Necklace Lie algebra

The *generalised necklace Lie algebra* associated to the dg vector space A is the Lie algebra $\mathfrak{neck}_A$ with the underlying vector space

$$\mathrm{s}\prod_{\substack{N\geqslant \mathbf{3}\\ N=n+m}}\left(\bigoplus_{1\leqslant m<N}\left(\bigoplus_{\lambda_{\mathbf{1}}+\cdots+\lambda_m=n} A\otimes((\mathrm{s}A)^*)^{\otimes\lambda_{\mathbf{1}}}\otimes\cdots\otimes A\otimes((\mathrm{s}A)^*)^{\otimes\lambda_m}\right)^{\mathbb{Z}/m\mathbb{Z}}\right)$$

Crucial point

The specific form of $\mathrm{s}A\oplus A^*$ implies that the Lie bracket of $\mathfrak{neck}_A$ splits into two.

$X * Y$ is the summand of $\{X, Y\}$ made up of the terms where one applies the linear pairing $\langle f, \mathrm{s}x\rangle$, where $f\in A^*$ comes from X and $x\in A$ comes from Y .

So the Lie bracket comes from the Lie-admissible product $*$:

$$\{X, Y\} = X * Y - (-1)^{|X||Y|} Y * X \ .$$

Curvature Necklace Lie algebra

The *curvature necklace Lie-admissible algebra* associated to the dg vector space A is $(\mathfrak{cneck}_A, d, *)$ with the underlying dg vector space

$$\mathrm{s} \prod_{\substack{N \geqslant 1 \\ N=n+m}} \left(\bigoplus_{1 \leqslant m < N} \left(\bigoplus_{\lambda_1+\cdots+\lambda_m=n} A \otimes ((\mathrm{s}A)^*)^{\otimes \lambda_1} \otimes \cdots \otimes A \otimes ((\mathrm{s}A)^*)^{\otimes \lambda_m} \right)^{\mathbb{Z}/m\mathbb{Z}} \right) .$$

Remark

The extension of the product from $N \geqslant 3$ to $N \geqslant 1$ corresponds to the surjection $c\mathscr{As}^{\mathrm{i}} \twoheadrightarrow \mathscr{As}^{\mathrm{i}}$ where $c\mathscr{As}^{\mathrm{i}}$

Definition (Almost pre-Calabi–Yau algebra)

A structure of an *almost pre-Calabi–Yau algebra* on a graded vector space A is a Maurer–Cartan element in the curved necklace Lie-admissible algebra

Higher Hochschild complex

Recall that there is a canonical inclusion

$$\begin{array}{rcl} W \otimes V^* & \hookrightarrow & \mathrm{Hom}(V, W) \\ x \otimes f & \mapsto & \big(v \mapsto xf(v)\big) \ , \end{array} \tag{1}$$

Definition (Higher Hochschild complex)

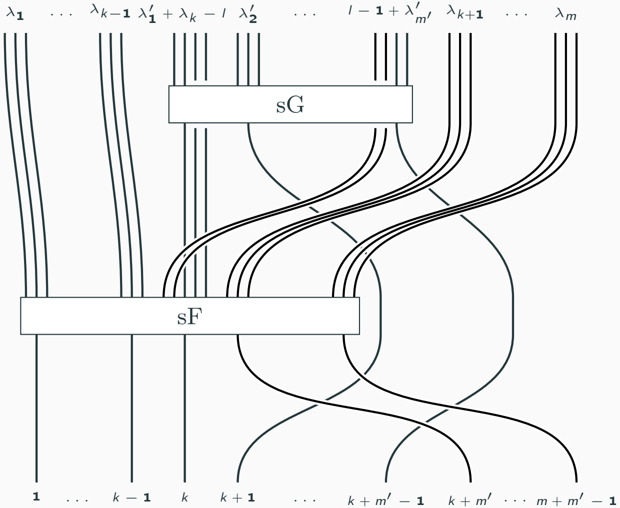
The *higher Hochschild complex* associated to a dg vector space A is the Lie-admissible algebra $\mathfrak{hhc}_A = (hhc, \partial\ , \circledast)$ where hhc is

$$\mathrm{s} \prod_{\substack{N \geqslant 1 \\ N=n+m}} \left(\bigoplus_{1 \leqslant m < N} \mathrm{Hom}_{\mathbb{Z}/m\mathbb{Z}} \left(\bigoplus_{\lambda_1 + \cdots + \lambda_m = n} \bigotimes_{j=1}^{m} (\mathrm{s}A)^{\otimes \lambda_j}, A^{\otimes m} \right) \right) \ ,$$

and $\mathrm{sF} \circledast \mathrm{sG} = \sum_{i=1}^{n} \mathrm{sF} \circledast_i^1 \mathrm{sG}$.

Lie admissible product of $\mathfrak{hhc}_A$: an illustration

$\mathrm{sF} \circledast^{1}_{\lambda_1+\cdots+\lambda_{k-1}+l} \mathrm{sG} =$



Pre-Calabi–Yau algebra

Proposition

The inclusion (1) induced the following inclusion of Lie admissible algebras

$$\mathfrak{cneck}_A \hookrightarrow \mathfrak{hhc}_A.$$

Definition (Pre-Calabi–Yau algebra)

A structure of *a pre-Calabi–Yau algebra* on a graded vector space A is a Maurer–Cartan element in the higher Hochschild Lie-admissible algebra $\mathfrak{hhc}_A$,

3 – Main theorem and consequences

Main theorem

Theorem (L.–Vallette)

For any dg vector space A, there is a canonical and functorial isomorphism of dg Lie-admissible algebras

$$\mathfrak{c}\mathfrak{D}\mathfrak{P}\mathfrak{ois}_A \cong \mathfrak{hhc}_A \ .$$

Remark

The letter $\mathfrak{c}$ of $\mathfrak{c}\mathfrak{D}\mathfrak{P}\mathfrak{ois}_A$ denotes the addition of a curvature in the double Poisson up to homotopy structure. Also, we have the embedding of Lie-admissible algebras $\mathfrak{D}\mathfrak{P}\mathfrak{ois}_A \hookrightarrow \mathfrak{c}\mathfrak{D}\mathfrak{P}\mathfrak{ois}_A$.

"Proof" of the main theorem – 1

Recall that

$$cDPois_A = \prod_{m \geqslant 1, n \geqslant 0} \mathrm{Hom}_{\mathfrak{S}_m^{\mathbf{op}} \times \mathfrak{S}_n} \left(\mathrm{cDPois}^{\mathrm{i}}(m,n), \mathrm{Hom}\left(A^{\otimes n}, A^{\otimes m}\right)\right),$$

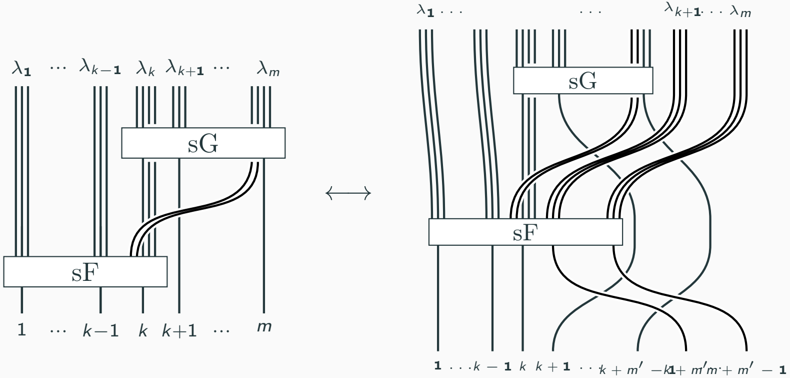
where $\mathrm{cDPois}^{\mathrm{i}}$ is cogenerated by $\nu_{\lambda_1,\ldots,\lambda_m}$ of degree $\lambda_1 + \cdots + \lambda_m - 1$

$$\nu_{\lambda_1,\ldots,\lambda_m} =$$

with $\nu_{\lambda_1,\ldots,\lambda_m} = \pm \nu_{\lambda_2,\ldots,\lambda_m,\lambda_1}$ then

$$cDPois_A \cong \mathrm{s} \prod_{\substack{N \geqslant 1 \\ N = n+m}} \left(\bigoplus_{1 \leqslant m < N} \mathrm{Hom}_{\mathbb{Z}/m\mathbb{Z}} \left(\bigoplus_{\sum \lambda_i = n} \bigotimes_{j=1}^{m} (\mathrm{s}A)^{\otimes \lambda_j}, A^{\otimes m} \right)\right) = hhc_A$$

"Proof" of the theorem – 2



$\mathrm{sF} \star \mathrm{sG}$ $\mathrm{sF} \circledast \mathrm{sG}$

The most difficult part of the proof is to check the signs.

Why this result is nice ?

The description of pre-Calabi–Yau structure in terms of properadic ones gives us a **notion of ∞-morphism between pre-Calabi–Yau algebras**, using [Hoffbeck–L.–Vallette 2020].

Remark

Kontsevich–Takeda–Vlassopoulos gave a notion of morphism between pre-Calabi–Yau algebras, but their first definition was "perfectible".

∞-morphism of DPois_∞-gebras

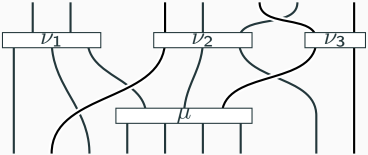
A ∞-morphism $f : (A, \alpha) \rightsquigarrow (B, \beta)$ of DPois_∞-gebras is a collection

$$\left\{ f_{s,e} : \mathrm{DPois}^{\text{i}}(s,e) \longrightarrow \mathrm{End}_B^A(s,e) = \mathrm{Hom}_k(A^{\otimes e}, B^{\otimes s}) \right\}_{s,e \in \mathbb{N}^*} .$$

which satisfies $\partial(f) = f \rhd \alpha - \beta \lhd f$, where

$$\beta \lhd f : \overline{\mathrm{DPois}}^{\text{i}} \xrightarrow{\Delta_{(*)}} \overline{\mathrm{DPois}}^{\text{i}} \lhd_{(*)} \mathrm{DPois}^{\text{i}} \xrightarrow{\beta \lhd_{(*)} f} \mathrm{End}_B \lhd_{(*)} \mathrm{End}_B^A \longrightarrow \mathrm{End}_B^A$$

$$f \rhd \alpha : \overline{\mathrm{DPois}}^{\text{i}} \xrightarrow{{}_{(*)}\Delta} \mathrm{DPois}^{\text{i}}{}_{(*)} \rhd \overline{\mathrm{DPois}}^{\text{i}} \xrightarrow{f\,{}_{(*)}\rhd \alpha} \mathrm{End}_B^A{}_{(*)} \rhd \mathrm{End}_A \longrightarrow \mathrm{End}_B^A$$



$\in \ \overline{\mathrm{DPois}}^{\text{i}} \lhd_{(*)} \mathrm{DPois}^{\text{i}}$

Composition of two ∞-morphisms

The *composite* of ∞-morphisms is defined by

$$g \circledcirc f \ : \ \mathrm{DPois}^{\mathsf{i}} \xrightarrow{\Delta} \mathrm{DPois}^{\mathsf{i}} \boxtimes \mathrm{DPois}^{\mathsf{i}} \xrightarrow{g \boxtimes f} \mathsf{End}_C^B \boxtimes \mathsf{End}_B^A \longrightarrow \mathsf{End}_C^A \ .$$

Proposition [Hoffbeck–L.–Vallette 2020]

Homotopy double Poisson gebras equipped with their ∞-morphisms and the composite $\circledcirc$ form a category.

∞-quasi-isomorphism

Definition (∞-quasi-isomorphim)

A ∞-morphism f is a ∞-quasi-isomorphism $f_{1,1}(I) : A \to B$ is a quasi-isomorphism of dg vector space.

Homotopy Transfer Theorem [Hoffbeck–L.–Vallette 2020]

For any contraction of dg vector space A

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

and any homotopy double Poisson gebra structure on A, there exists homotopy double Poisson gebra structure on H and extensions of the chain maps i and p into ∞-quasi-isomorphisms.

∞-quasi-isomorphism

Theorem [Hoffbeck–L.–Vallette 2022 ?]

Two homotopy double Poisson gebra structures (A, α) and (B, β) are ∞-quasi-isomorphic if and only if they are related by a zig-zag of (strict) quasi-isomorphisms of homotopy double Poisson gebras:

$$\begin{matrix} \exists\ \infty\text{-quasi-isomorphism} \\ (A, \alpha) \overset{\sim}{\rightsquigarrow} (B, \beta) \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} \exists \text{ zig-zag of quasi-isomorphisms} \\ (A, \alpha) \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \cdots\cdots \bullet \xleftarrow{\sim} \bullet \xrightarrow{\sim} (B, \beta) \end{matrix} \quad .$$

Extension to pre-Calaby–Yau algebras

All these notions/results extend to pre-Calabi–Yau algebras under assumption of completeness of the underlying dg vector space for a degree-wise decreasing filtration.

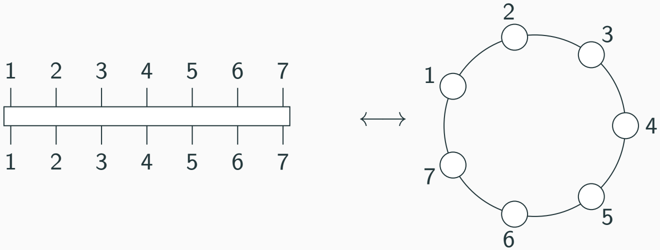
Description of the coproduct of $\mathrm{DPois}^{\mathrm{i}}$: a new combinatorics

Recall that

the coproduct of $\mathrm{DPois}^{\mathrm{i}}$ is difficult to describe ...

... but we give a new underlying combinatoric for this description.

As $\mathrm{DPois}^{\mathrm{i}} \cong \mathrm{DLie}^{\mathrm{i}} \boxtimes \mathrm{Ass}^{\mathrm{i}}$, *one can just describe the coproduct of* $\mathrm{DLie}^{\mathrm{i}}$*.*
We encode the cyclic symmetry in the combinatorics.



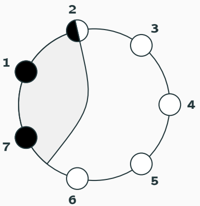
Elementary coloured cutting

Definition

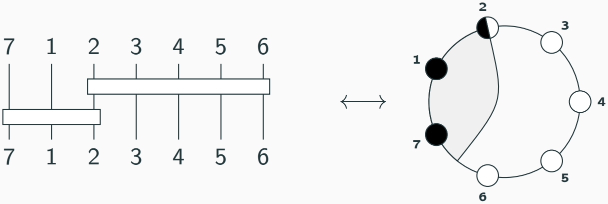
An *elementary coloured cutting* of a bangle is defined by the following two-steps construction.

Cutting choose a bead of the bangle and cut the bangle into two parts, that is draw a line starting from the bead, splitting it into two, to an edge between two beads, such that each half-bangle contains at least one bead.

Colouring colour the beads on the clockwise side of the half-bean in white and the other ones, as well as the entire sector, in black.

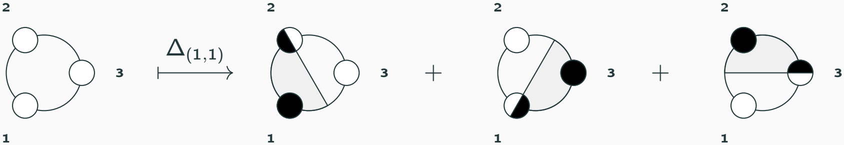


New combinatorics for partial coproduct of $\mathrm{DLie}^{¡}$



Proposition [L.–Vallette]

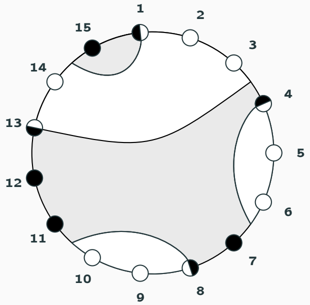
The terms appearing in the infinitesimal decomposition map of the coproperad $\mathrm{DLie}^{¡}$ are in one-to-one correspondance with the elementary coloured cuttings of bangles.



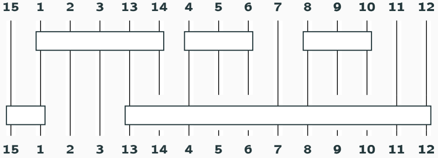
New combinatorics for coproduct of DLie^{i}

Proposition [L.–Vallette]

The various terms appearing in the decomposition map of the coproperad DLie^{i} are in one-to-one correspondance with partitioned bangles.



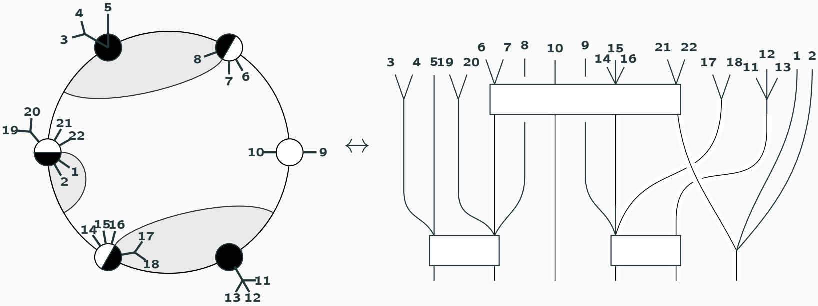
$\leftrightarrow$



New combinatorics for coproduct of $\mathrm{DPois}^{\text{¡}}$

Proposition [L.–Vallette]

The various terms appearing in the decomposition map of the coproperad $\mathrm{DPois}^{\text{¡}}$ are in one-to-one correspondance with hairy partitioned bangles.



Thanks for your attention.