HOMOTOPY GROUPS OF SOME EMBEDDING SPACES

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Based on the joint work with Peter Teichner (MPIM Bonn)
https://arxiv.org/abs/2105.13032

Table of contents

Motivation

Spaces of embeddings

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- **General goal.** Study the homotopy type of the space

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of smooth neat embeddings $K \colon V \hookrightarrow M$ which near ∂V agree with a fixed basepoint $\phi \colon V \hookrightarrow M$. We denote $\mathsf{s} := \phi|_{\partial V} \colon \partial V \hookrightarrow \partial M$.

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- Recall that a smooth map $K\colon V \to M$ is an embedding if it is *injective* and at any $v \in V$ the derivative $dK|_v$ is *injective*, and K is neat if it is transverse to the boundary and $K(V) \cap \partial M = K(\partial V)$.

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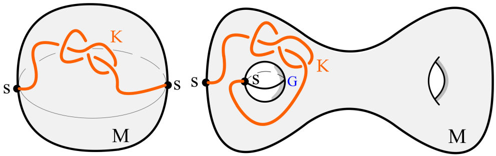

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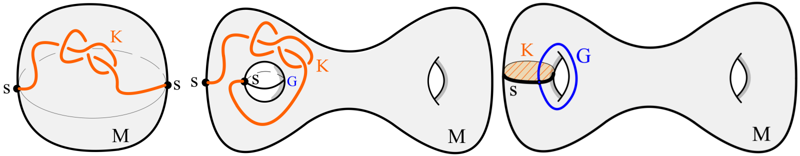

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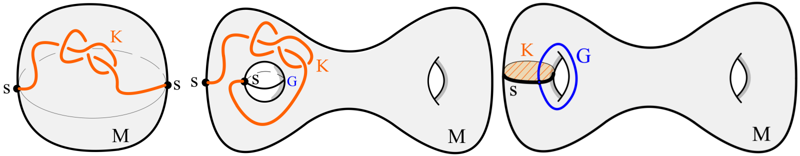

- For $V = \mathbb{D}^k$, the setting with a dual: if there exists $G \colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that G has trivial normal bundle and $G \pitchfork \mathbf{s} = \{pt\}$. Like pictures 2 and 3!

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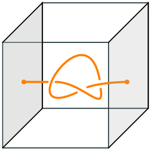

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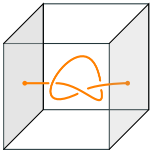

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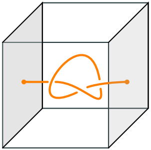

- Recently, intensively studied is the set of (long) 2-knots in a 4-manifold M:

$${}_0\,\mathrm{Emb}_{@}(\mathrm{D}^2; M)$$

This can be huge – for example, "spinning" a classical knot gives a 2-knot in $_0\,\mathrm{Emb}_{@}(\mathrm{S}^2; \mathrm{R}^4) = {}_0\,\mathrm{Emb}_{@}(\mathrm{D}^2; \mathrm{D}^4)$.

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For any $1 \leq k \leq d$, in a setting with a dual,

Recall that setting with a dual means: we have a d-manifold M and embedding $\mathsf{s} = \partial \mathbb{U}\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$, such that there exists $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$ with trivial normal bundle and such that $G \pitchfork \mathsf{s} = \{pt\}$.

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Theorem 1 (Space level light bulb trick [K–Teichner '21])

For any $1 \leq k \leq d$, in a setting with a dual, any choice of $\Psi : \mathbb{D}^k \hookrightarrow M$ leads to an (explicit) homotopy equivalence

$$\mathrm{Emb}_\partial(\mathbb{D}^k; M) \simeq \mathrm{Emb}^{\varepsilon}_\partial(\mathbb{D}^{k-1}; X),$$

where $X := M \cup_G h^{d-k+1}$.

Superscript ε means embedded disks are equipped with “push-offs”...

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where $X := M \cup_G h^{d-k+1}$. In particular, if $d = 4$ we have

$$\pi_0\,\mathrm{Emb}_\partial(D^2; M) = \pi_1\,\mathrm{Emb}^\varepsilon_\partial(D^1; X).$$

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The main result today, and applications

- Note. $\dim X \quad \dim \mathrm{D}^{k \quad 1} > \dim M \quad \dim \mathrm{D}^{k}$

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Theorem 2 [K–Teichner ’22]

Fix $`; d \quad 1$ such that $d \quad ` + 3$ and $d \quad 2` \quad 1$. Let $X$ be a smooth compact $d$-dimensional manifold with boundary, and fix $u \colon \mathrm{D}^{`} \, ,\! \quad X$. Then

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$$\frac{\mathbb{Z}[\pi_1 X]}{\langle 1 \rangle + rel_{\ell,d}} \cong \mathrm{dax}(\pi_{d-\ell}(X)) \underset{\mathrm{Dax}}{\overset{\partial r}{\rightleftarrows}} \pi_{d-2\ell-1}(\mathrm{Emb}_{\partial}(\mathbb{D}^\ell; X); u) \xrightarrow{p_u} \pi_{d-\ell-1}X.$$

where the invariant Dax is defined on the image of the realisation map ∂r and is its explicit inverse, and $rel_{1,d} := \emptyset$ and $rel_{\ell,d} := \langle g - (-1)^{d-\ell} g^{-1} : g \in \pi_1 X \rangle$ if $\ell \geq 2$.

How to compute homotopy groups?

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- We make this more explicit, and compute many classes of examples in K' 21.

Applications of the two theorems

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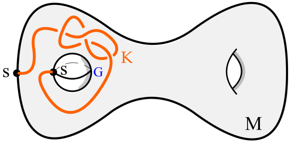

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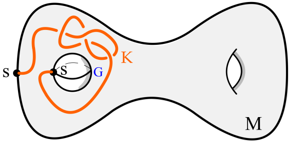

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$\implies$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

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Applications of the two theorems

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- $\Rightarrow$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.
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  - Moreover, we get an (unexpected) group structure on $\pi_0\,\mathrm{Emb}_\partial(\mathrm{D}^2; M)$! It is usually nonabelian!

$k = d-1:\ \mathrm{Emb}_\partial(\mathrm{D}^{d-1}; \mathrm{S}^1 \times \mathrm{D}^{d-1}) \simeq \Omega\,\mathrm{Emb}_\partial(\mathrm{D}^{d-2}; \mathrm{D}^d)$

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Open problem

Is $\pi_0\,\mathrm{Diff}^+_\partial(\mathrm{D}^4)$ trivial? Compute it.

See Budney-Gabai, Gay, Watanabe for some candidate diffeomorphisms.

Metastable homotopy groups

Stable, metastable, meta^{2}stable...(?)

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  - $\mathrm{Emb}(V;X) \to \mathrm{P}_n(V;X)$ is $(nd - (n+1)\ell - (2n-1))$-connected (hard!).
  - Use homotopy theoretic tools to study $\mathrm{P}_n(V;X)$.

- Therefore, part 1) in Theorem 2, which said

  $$p_u\colon \pi_n(\mathrm{Emb}_\partial(\mathrm{D}^\ell;X);u) \cong \pi_n(\mathrm{Imm}_\partial(\mathrm{D}^\ell;X);u) \cong \pi_{n+\ell}X, \quad \text{for } 0 \le n \le d-2\ell-2.$$

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About the lowest degree in the metastable range

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    It turns out this is given as the image of a certain homomorphism

    $$\mathrm{dax} : \pi_{d-\ell}X \to \mathbb{Z}[\pi_1 X \setminus 1].$$

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About the lowest degree in the metastable range

Theorem [Dax '72]

There is an isomorphism ${}_{d\quad 2`\quad 1}(\mathrm{Imm}(V;X);\mathrm{Emb}(V;X);u) = {}_0(C_u;\ {}_u)$, the degree 0 normal bordism group of a certain space C_u with a stable normal bundle ${}_u$ over it.

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i.e. F is smooth and its track

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The realisation map and the Dax invariant

- Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial \mathrm{r}(g)$ is given by

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- We can compute this in many classes of examples! See [K '21].

Thank you!