Hochschild cohomology of Intersection algebra

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Réunion annuelle du GDR de Topologie algébrique
Nantes - october 2022

Main theorem

Theorem [R22]

Let X be a closed, connected, oriented **pseudomanifold**.
The **Hochschild cohomology** of the **blown-up complex** $\tilde{N}^*_{\bullet}(X)$

$$\left(HH^*(\tilde{N}^*_{\bullet}(X), \tilde{N}^*_{\bullet}(X)), \cup, [-,-], \Delta\right)$$

is a **Batalin-Vilkovisky algebra**.

Outline

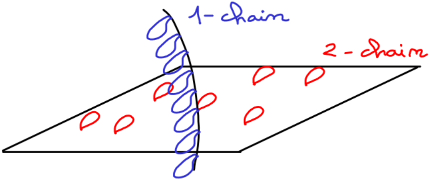
BV algebras

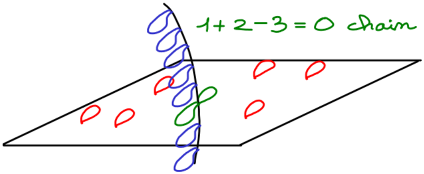
Loop homology

M m-manifold. $\mathcal{L}M = \mathcal{C}^0(\mathbb{S}^1, M)$. $\quad M \underset{ev_0}{\overset{cst}{\rightleftarrows}} \mathcal{L}M$

loop product

$$\bullet : H_i(\mathcal{L}M) \otimes H_j(\mathcal{L}M) \to H_{i+j-m}(\mathcal{L}M)$$





Loop homology

M m-manifold. $\mathcal{L}M = \mathcal{C}^0(\mathbb{S}^1, M)$. $\quad M \underset{ev_0}{\overset{cst}{\rightleftarrows}} \mathcal{L}M$

loop product

$$\bullet : H_i(\mathcal{L}M) \otimes H_j(\mathcal{L}M) \to H_{i+j-m}(\mathcal{L}M)$$

loop bracket

$$\{-,-\} : H_i(\mathcal{L}M) \otimes H_j(\mathcal{L}M) \to H_{i+j+1-m}(\mathcal{L}M)$$

with $\{a, b\} = a * b - (-1)^{(|a|+1)(|b|+1)} b * a$

and operator Δ given by $\mathbb{S}^1$ action. Reindex $\mathbb{H}_* = H_{*+m}(\mathcal{L}M)$

String topology

Theoreom (Chas-Sullivan [CS99])

$(\mathbb{H}_*, \bullet, \{-,-\})$ a **Gerstenhaber algebra**:

- $\bullet$ is graded commutative and associative,
- $\{-,-\}$ is a Lie bracket of degree 1: $\forall a, b, c \in \mathbb{H}_*$
  - $\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\}$
  - $\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}$
- $\{a, b\bullet c\} = \{a,b\}\bullet c + (-1)^{|b|(|a|-1)} b \bullet \{a,c\}$.

Theorem [CS99]

$(\mathbb{H}_*, \bullet, \{-,-\}, \Delta)$ is a **Batalin Vilkovisky algebra**:

- $\Delta \circ \Delta = 0$,
- $(-1)^{|a|}\{a,b\} = \Delta(a\bullet b) - \Delta a \bullet b - (-1)^{|a|} a \bullet \Delta b$.

Hochschild cohomology I

Definition

The **Hochschild cohomology** of an algebra A is

$$HH^*(A,A) := Ext^*_{A^e}(A,A) = H_* Hom_{A^e}(P,A)$$

with $P \to A$ a cofibrant resolution of A.

Theorem (Gerstenhaber [Ger63], Getzler [Get93])

The Hochschild cohomology $(HH^*(A,A), \cup, [-,-])$ of a DGA A is a **Gerstenhaber algebra**.

Hochschild cohomology II

Theorem (Menichi [Men09])

M compact, connected, oriented smooth $m-$manifold.

$$(HH^*(C^*(M), C^*(M)), \cup, [-,-], \Delta)$$

is a **Batalin-Vilkovisky algebra**.

Δ on $HH^*(C^*(M), C^*(M)^\vee)$

$$HH^*(C^*(M), C^*(M)^\vee) \simeq HH^*(C^*(M), C^*(M))$$

by **derived Poincaré duality**: we have quasi-isomorphisms

$$(C^*(M))^\vee \xleftarrow{\simeq} P \xrightarrow{\simeq} C^*(M)$$

with P a cofibrant approximation of $C^*(M)$.

Theorem ([CJ02], [Mer04], [FTV07])

We have a BV-algebra isomorphism

$$HH^*(C^*(M), C^*(M)) \simeq \mathbb{H}_*(\mathcal{L}M).$$

Poincaré duality

Different approaches

Two approaches to Poincaré duality:

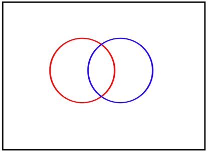
1. geometrical: via intersection product
2. structural: cup & cap products

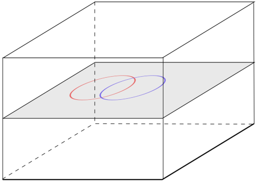
Transversality

Definition

M manifold, N & P submanifolds are **transverse** if

$$\forall x \in N \cap P, \quad T_x M = T_x N + T_x P.$$





Proposition

$$\dim(N \pitchfork P) = \dim(N) + \dim(P) - \dim(M).$$

Poincaré Duality

intersection product: $\pitchfork : H_i(M) \otimes H_j(M) \to H_{i+j-m}(M)$.

Proposition

M is a closed, oriented, smooth $m-$manifold. The bilinear form

$$H_i(M; \mathbb{Q}) \otimes H_{m-i}(M; \mathbb{Q}) \to H_0(M; \mathbb{Q}) \to \mathbb{Q}$$

is non degenerate.

Theorem (Poincaré duality)

For $0 \leq i \leq m$

$$H_{m-i}(M; \mathbb{Q}) \simeq Hom(H_i(M; \mathbb{Q}); \mathbb{Q}) \simeq H^i(M; \mathbb{Q}).$$

Chain-level intersection product

$\pitchfork: C_*(M) \otimes C_*(M) \to C_*(M)$ partial product. $C_*^{\pitchfork}(M)$ its domain.

Theorem (McClure [McC06])

M compact, oriented, PL manifold. The inclusion

$$C_*^{\pitchfork} \hookrightarrow C_*^{PL}(M) \otimes C_*^{PL}(M)$$

is a quasi-isomorphism.

Proposition

The intersection product in homology is induced by the composite

$$C_*^{PL}(M) \otimes C_*^{PL}(M) \xleftarrow{\simeq} C_*^{\pitchfork}(M) \to C_*^{PL}(M).$$

Cup and Cap products

We have cup and cap which are defined at the (co)chain level.

$$\cup : C^p(M) \otimes C^q(M) \to C^{p+q}(M)$$

$$\cap : C^p(M) \otimes C_q(M) \to C_{q-p}(M)$$

Proposition

$(C^*(M), \cup)$ is an algebra. $C_*(M)$ is a $C^*(M)$ module.

Theorem (Poincaré duality)

We have an isomorphism of algebras

$$(H^*(M), \cup) \xrightarrow[\simeq]{-\cap[M]} (H_{*-m}(M), \pitchfork).$$

Back to Hochschild cohomology

M compact, connected, oriented smooth $m-$manifold.

$$\begin{array}{ccc} H_*(\mathcal{L}M) & \xrightarrow[\simeq]{J} & HH^*(C^*(M), C^*(M)^\vee) \\ H(ev_0) \searrow \nwarrow H(cst) & & \swarrow HH^*(\eta, C^*(M)^\vee) \\ & H_*(M) \simeq H(C^*(M)^\vee) & \end{array}$$

with **Jones isomorphism** J [Jon87].

The unit $\eta : \mathbb{Q} \to A$ induces $HH^*(\eta, C^*(M)^\vee)$

$$HH^*(C^*(M), C^*(M)^\vee) \to HH^*(\mathbb{Q}, C^*(M)^\vee) \simeq H(C^*(M)^\vee).$$

$J \circ H(cst)([M])$ gives a morphism $P \to C^*(M)^\vee$.
By Poincaré duality we can show that it is a quasi-isomorphism.

Derived Poincaré duality algebra

We have quasi-isomorphisms

$$(C^*(M))^\vee \xleftarrow{\simeq} P \xrightarrow{\simeq} C^*(M)$$

with P a cofibrant approximation of $C^*(M)$.

In other words,

$$C^*(M) \simeq (C^*(M))^\vee \text{ in } D(C^*\text{-bimodules}).$$

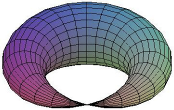
We say that $C^*(M)$ is a **Derived Poincaré duality algebra**.

Recap

M compact, connected, oriented, smooth $m-$manifold.

(co)chain complex	$C^*(M)$	algebra for $\cup$
	$C_*(M)$	$C^*(M)$-module for $\cap$
Poincaré duality	$C^*(M) \xrightarrow[\simeq]{-\cap[M]} C_*(M)$	
Geometric Poincaré duality	$H_*(M;\mathbb{Q}) \otimes H_{m-*}(M;\mathbb{Q}) \to \mathbb{Q}$	
Derived Poincaré duality	$(C^*(M))^\vee \overset{\simeq}{\leftarrow} P \overset{\simeq}{\rightarrow} C^*(M)$	

What about spaces with singularities ?

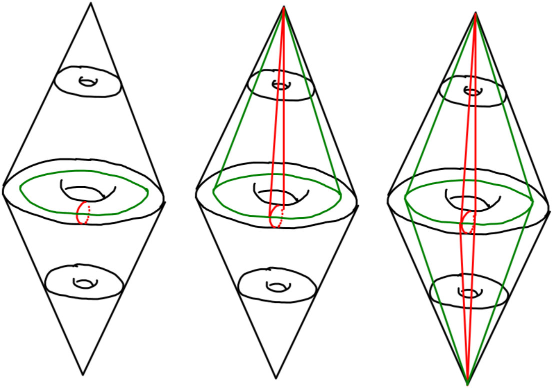


- No Poincaré duality
- Goresky, McPherson [GM80]: intersection cohomology $I_{\bar{\bullet}}H^*$
- Chataur, Saralegui, Tanré [CST20]: blown up complex $\tilde{N}^*_{\bar{\bullet}}$

Intersection homology

Failure of Poincaré duality I

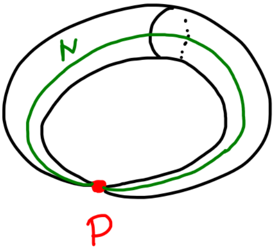
$X = S\mathbb{T}^2$



$$H_3(X,\mathbb{Q}) = \mathbb{Q}$$
$$H_2(X,\mathbb{Q}) = \mathbb{Q}\oplus\mathbb{Q}$$
$$H_1(X,\mathbb{Q}) = 0$$
$$H_0(X,\mathbb{Q}) = \mathbb{Q}$$

Poincaré duality fails since $H_1(X,\mathbb{Q}) \not\simeq H_2(X,\mathbb{Q})$.
Need to control how chain intersect singularities.

Failure of Poincaré duality II



$\dim(N \cap P) = 0$ and $\dim(N) + \dim(P) - \dim(X) + \varepsilon = 0 + 1 - 2 = -1$

$$\dim(N \cap P) \neq \dim(N) + \dim(P) - \dim(X)$$

Solution: add some margin of error on singular part.

Pseudomanifolds

Definition

filtered space X of **formal dimension** n

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

$\mathcal{S}_i$, i-**strata** of X: connected components of $X_i \setminus X_{i-1}$.
$\mathcal{S}_X = \{\text{strata of } X\}$, **regular** strata: n-strata, **singular** strata

Definition

X **pseudomanifold**

- X filtered space
- i-strata are i-manifolds
- locally cone like

Example: $\mathbb{S}^2 \vee \mathbb{S}^2$

Intersection chains

Definition

A map $\overline{p} : \mathcal{S}_X \to \mathbb{Z}$ such that $\overline{p}(\mathcal{S}_n) = 0$ is called a **perversity**.

Definition

A singular chain of degree i, σ is of **$\overline{p}$-intersection** if

- $\dim(\sigma \cap S) \leq i + \dim(S) - f\dim(X) + \overline{p}(S)$,
- $\dim(\partial\sigma \cap S) \leq i - 1 + \dim(S) - f\dim(X) + \overline{p}(S) \quad \forall S \in \mathcal{S}_X$.

$\overline{p}$-intersection chain complex $I^{\overline{p}}C_*$
$\overline{p}$-intersection homology $I^{\overline{p}}H_*$

Dually, we have **$\overline{p}$-intersection cochain complex** $I_{\overline{p}}C^*$.

Properties of singular homology

Properties (stratified Eilenberg-Steerod axioms)

- Invariance under stratified homotopy
- Cone formula
- Mayer-Vietoris sequences

Theorem (Goresky-McPherson-Poincaré duality [GM80])

If X is a connected, oriented, closed n-pseudomanifold then, we have a non degenerate bilinear form

$$\pitchfork\colon I^{\overline{p}}H_i(X;\mathbb{Q}) \otimes I^{D\overline{p}}H_{n-i}(X;\mathbb{Q}) \to \mathbb{Q}.$$

Not natural, not satisfied for any ring

Blown-up intersection cochains

Blown-up cochain complex

$$\tilde{N}^*_\bullet(\_; R) : \mathcal{PM} \to pDGA$$

Properties

- computes $I_{\overline{p}}H^*$
- cup and cap products defined at (co)chain level
- Poincaré duality: $\tilde{N}^*_{\overline{p}}(X;R) \xrightarrow[\simeq]{\cap[X]} I^{\overline{p}}C_{n-*}(X;R)$

Recap

X is a connected, oriented, closed n-pseudomanifold.

(co)chain complex	$\tilde{N}^*_{\bar{\bullet}}(X)$	algebra for $\cup$
	$I^{\bar{\bullet}}C_*(X)$	$\tilde{N}^*_{\bar{\bullet}}(X)$-module for $\cap$
Poincaré duality	$\tilde{N}^*_{\bar{\bullet}}(X) \xrightarrow[\simeq]{-\cap[X]} I^{\bar{\bullet}}C_*(X)$	
Geometric Poincaré duality	$\pitchfork\colon I^{\bar{\bullet}}H_*(X;\mathbb{Q}) \otimes I^{D\bar{\bullet}}H_{n-*}(X;\mathbb{Q}) \to \mathbb{Q}$	
Derived Poincaré duality	$(\tilde{N}^*_{\bar{\bullet}}(X))^\vee \xleftarrow{\simeq} B(\tilde{N}^*_{\bar{\bullet}}(X)) \xrightarrow{\simeq} \tilde{N}^*_{\bar{\bullet}}(X)$	

Goals

Define Hochschild cohomology for $\tilde{N}^*_{\bar{\bullet}}$, find algebraic structures, interpret them topologically.

Hochschild (co)homology for pDGA

Perverse objects

$\tilde{N}^*_{\bullet}(X)$ is a **perverse cochain complex**

$$\tilde{N}^*_{\bullet}(X) : Perv_{\leq} \to Ch(R)$$

Proposition

Let $(\mathcal{C}, \boxtimes_{\mathcal{C}}, Hom_{\mathcal{C}}, \mathcal{I})$ be a closed symmetric monoidal category then the category of **perverse objects** on $\mathcal{C}$ is also closed symmetric monoidal $(\mathcal{C}^{Perv}, \boxtimes, Hom_{\mathcal{C}^{Perv}}, \mathcal{I}_{\mathcal{C}^{Perv}})$.

$\tilde{N}^*_{\bullet}(X)$ is a **perverse differential graded algebra** (pDGA).

$$\begin{array}{ccccc} & & \tilde{N} & & \\ & \overset{\cup}{\nearrow} & & \overset{\cup}{\nwarrow} & \\ \tilde{N} \boxtimes \tilde{N} & & & & \tilde{N} \boxtimes \tilde{N} \\ \uparrow{\scriptstyle id \boxtimes \cup} & & & & {\scriptstyle \cup \boxtimes id}\uparrow \\ \tilde{N} \boxtimes (\tilde{N} \boxtimes \tilde{N}) & & \xleftarrow{\quad a \quad} & & (\tilde{N} \boxtimes \tilde{N}) \boxtimes \tilde{N} \end{array}$$

Hochschild (co)homology

Bar construction: $B(A) = \oplus_{k\in\mathbb{N}} A \boxtimes A^{\boxtimes k} \boxtimes A = A \boxtimes T(A) \boxtimes A$.

Definition

A pDGA and N A-bimodule, the **Hochschild (co)chains** are given

$$\text{by } HC_*^{\bar{\bullet}}(A,N) := N \boxtimes_{A^e} B(A) \simeq N \boxtimes T(A),$$

$$HC^*_{\bullet}(A,N) := Hom_{A^e}(B(A),N) \simeq Hom_{(Ch(R))^{Perv}}(T(A),N).$$

where $A^e = A \boxtimes A^{op}$.

Derived functors

Theorem ([Hov09])

- We have a model category structure on $Ch(R)^{Perv}$: a morphism is a weak equivalence (or a fibration) if and only if it so perversity wise.
- We have a model category structure on A-modules.

Definition

$$Tor_*^{A^e}(N_1, N_2) = H_*(P \boxtimes_{A^e} N_2)$$

$$Ext^*_{A^e}(N_1, N_2) = H_* Hom_{A^e}(P, N_2)$$

with $P \to N_1$ a cofibrant resolution of N_1.

Equivalent definitions

Theorem ([BMR13], [R22])

Let M be a pDG A-modules, the following statements are equivalent.

- M is cofibrant,
- M is semi-projective,
- M is a retract of semi-free pDG A-module.

Proposition

$B(A)$ is a cofibrant resolution of A if

- A is cofibrant in $Ch(R)^{Perv}$,
- or R is a field.

Algebraic structures I

Theorem [R22]

The Hochschild cohomology $(HH^*_\bullet(A,A), \cup, [-,-])$ of a pDGA A is a **Gerstenhaber algebra**. We have

- **cup product**: $-\cup - : HH^r_{\overline{p}} \boxtimes HH^s_{\overline{q}} \to HH^{r+s}_{\overline{p}+\overline{q}}$
- **bracket**: $[-,-] : HH^r_{\overline{p}} \boxtimes HH^s_{\overline{q}} \to HH^{r+s+1}_{\overline{p}+\overline{q}}$

$$f \cup g[a_1|\dots|a_k] = \sum_{i=1}^{k-1} \pm f[a_1|\dots|a_i]g[a_{i+1}|\dots|a_k]$$

$$[f,g] = f \circ g - (-1)^{(r+1)(s+1)} g \circ f$$

with

$$f \circ g[a_1|\dots|a_k] = \sum_{1\leq i<j\leq k} \pm f[a_1|\dots|a_i|g[a_{i+1}|\dots|a_j]|a_{j+1}|\dots|a_k].$$

Algebraic structures II

Theorem [R22]

The Hochschild cohomology

$$\left(HH^*(\tilde{N}^*_{\bar{\bullet}}(X), \tilde{N}^*_{\bar{\bullet}}(X)), \cup, [-,-], \Delta\right)$$

is a Batalin-Vilkovisky algebra.

On $HH_*(\tilde{N}^*_{\bar{\bullet}}(X), \tilde{N}^*_{\bar{\bullet}}(X))$, **Connes boundary**

$$bC(a_0[a_1|\ldots|a_k]) = \sum_{i=0}^{k} \pm 1[a_i|\ldots|a_n|a_0|\ldots|a_{i-1}]$$

On $\left(HH_*(\tilde{N}^*_{\bar{\bullet}}(X), \tilde{N}^*_{\bar{\bullet}}(X))\right)^\vee \simeq HH^*(\tilde{N}^*_{\bar{\bullet}}(X), \tilde{N}^*_{\bar{\bullet}}(X)^\vee)$

$$(bC^\vee f)(a_0[a_1|\ldots|a_k]) = (-1)^{|f|} \sum_{i=0}^{k} \pm f(1[a_i|\ldots|a_n|a_0|\ldots|a_{i-1}]).$$

Work in progress

- Spectral sequence to compute HH^*,
- Topological interpretation of algebraic structures.

Thank you for your attention.

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