Exercice sheet 1: Homological and Lie algebra

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Exercice 1: Show that if there exists a contracting homotopy of the complex (C^*, d_*), then its cohomology vanishes, i.e. $H^p(C^*, d_*) = 0$ for all $p \geq 0$.

Exercice 2: Show that a short exact sequence of complexes

$$0 \to C_1^* \xrightarrow{\varphi^*} C_2^* \xrightarrow{\psi^*} C_3^* \to 0$$

induces a long exact sequence in cohomology, i.e. there exists a k-linear map $\partial^p : H^p(C_3^*) \to H^{p+1}(C_1^*)$, called the connecting homomorphism, such that the sequence of k-vector spaces

$$\ldots \xrightarrow{\partial^{p-1}} H^p(C_1^*) \xrightarrow{H^p(\varphi)} H^p(C_2^*) \xrightarrow{H^p(\psi)} H^p(C_3^*) \xrightarrow{\partial^p} H^{p+1}(C_1^*) \to \ldots$$

is exact.

Exercice 3: Suppose there is a commutative diagram of k-vector spaces

$$\begin{tikzcd}
0 \arrow[r] & V' \arrow[r, "i"] \arrow[d, "\varphi'"] & V \arrow[r, "\pi"] \arrow[d, "\varphi"] & V'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & W' \arrow[r, "i'"] & W \arrow[r, "\pi'"] & W'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows and such that φ' and φ'' are isomorphisms. Show that φ is an isomorphism.

Exercice 4: Show that the kernel of a homomorphism of Lie algebras is an ideal. Construct a Lie algebra structure on the quotient vector space of a Lie algebra by an ideal. Have an example of a subalgebra of a Lie algebra which is not an ideal.

Exercice 5: Show that k is a $\mathfrak{g}$-module for any Lie algebra $\mathfrak{g}$. Show that $\mathfrak{g}$ is a $\mathfrak{g}$-module for any Lie algebra $\mathfrak{g}$ with respect to the "adjoint action", i.e. $x \cdot y := \mathrm{ad}_x(y) := [x, y]$ for all $x, y \in \mathfrak{g}$.

Exercice 6: Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. Show that the bracket defined by

$$[(a, x), (b, y)] := (x \cdot b - y \cdot a, [x, y]),$$

renders the vector space $V \oplus \mathfrak{g}$ a Lie algebra.