Exercice sheet 2: Cohomology of Lie algebras

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Exercice 1: Show the Cartan relations in on the Chevalley-Eilenberg complex:

(a) $L_x = d \circ i_x + i_x \circ d$,

(b) $L_x \circ L_y - L_y \circ L_x = L_{[x,y]}$,

(c) $L_x \circ i_y - i_y \circ L_x = i_{[x,y]}$,

(d) $L_x \circ d = d \circ L_x$.

Deduce that a Lie algebra acts trivially on its cohomology.

Exercice 2: Show that the derivations of a Lie algebra $\mathfrak{g}$ fit into an exact sequence:

$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \stackrel{\mathrm{ad}}{\to} \mathfrak{der}\,(\mathfrak{g}) \to \mathfrak{out}\,(\mathfrak{g}) \to 0.$$

Exercice 3:

(a) Show that a homomorphism of $\mathfrak{g}$-modules $f : V \to W$ induces a k-linear map

$$f_* : H^*(\mathfrak{g}, V) \to H^*(\mathfrak{g}, W), \;\; [c] \mapsto [f \circ c].$$

(b) Show that a short exact sequence of $\mathfrak{g}$-modules

$$0 \to V' \stackrel{f}{\to} V \stackrel{g}{\to} V'' \to 0$$

induces a long exact sequence in cohomology.

Exercice 4: Compute the cohomology of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Compute the cohomology of the abelian Lie algebra k^n. Compute the cohomology of the Heisenberg Lie algebra.