Exercice sheet 3: More cohomology

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Exercice 1: Let $L\mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$ be the loop algebra over the simple complex Lie algebra $\mathfrak{g}$. Such a Lie algebra admits an invariant scalar product $\langle , \rangle$. The bracket in $L\mathfrak{g}$ is given by the bracket in $\mathfrak{g}$. It possesses a central extension $\widehat{L\mathfrak{g}}$ given by the cocycle

$$\alpha(f, g) = \int_0^1 \langle f, dg \rangle.$$

Examine the crossed module

$$0 \to \mathbb{C} \to \widehat{L\mathfrak{g}} \to \mathfrak{Der}\,(\widehat{L\mathfrak{g}}) \to \mathrm{Vect}(S^1) \to 0.$$

Is it non-trivial ?

Exercice 2: W_1 is the Lie algebra generated by elements e_n with the bracket $[e_n, e_m] = (m-n)e_{n+m}$ for all $n, m \in \mathbb{Z}$, $n, m \geq -1$. Take as cochain spaces for W_1 the polynomial cochains

$$C^p(W_1, k) = \bigoplus_{l \in \mathbb{Z}} \bigoplus_{\substack{i_1 + \ldots + i_p = l \\ i_j \geq -1}} k\ \epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_p},$$

where ϵ_i is the element dual to e_i, i.e. $\epsilon_i(e_j) = \delta_{i,j}$.

(a) Compute the Lie derivative L_{e_0} on a cochain $\epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_p}$. Show that the subcomplex of all cochains $\epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_p}$ with non-zero eigenvalue under the action of L_{e_0} admits a contracting homotopy.

(b) Compute the subcomplex of cochains whose eigenvalue under L_{e_0} is zero. Use it to compute the cohomology of W_1.

(c) $k = \mathbb{C}$: Compare to the cohomology of $\mathfrak{sl}_2(\mathbb{C})$: Show that $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the subalgebra of W_1 generated by e_{-1}, e_0 and e_1.