Habilitation

Friedrich Wagemann
Université de Nantes

September 18, 2007

Abstract

This is a survey of the results constituting my habilitation thesis. It is based on the following articles.

Articles constituting the habilitation thesis

[30] F. Wagemann, Some Remarks on the Cohomology of Krichever-Novikov Algebras. Lett. Math. Phys. **47** (1999) 173–177

[31] F. Wagemann, Differential Graded Cohomology and Lie Algebras of Holomorphic Vector Fields. Comm. Math. Phys. **208** (1999) 521–540

[32] F. Wagemann, A Two Dimensional Analogue of the Virasoro Algebra. J. Geom. Phys **36** (2000) 103–116

[33] F. Wagemann, A Crossed Module representing the Godbillon-Vey Cocycle. Lett. Math. Phys. **51** (2000) 293–299

[34] F. Wagemann, Explicit Formulae for Cocycles of Holomorphic Vector fields with values in λ-Densities. J. Lie Theory, Vol. **11**, 1 (2001) 173–184

[8] A. Frabetti, F. Wagemann, On the Leibniz cohomology of vector fields. Ann. Global Anal. Geom. **21**, No. 2 (2002) 177–190

[23] K.-H. Neeb, F. Wagemann, Universal central extension of holomorphic current algebras. Manuscripta Math. **112** (2003) 441–458

[35] F. Wagemann, On Lie algebra crossed modules. Comm. Algebra **34** (2006) no. 5, 1699-1722 (math.KT/0611375)

[24] K.-H. Neeb, F. Wagemann, The second cohomology of current algebras of general Lie algebras. accepted at Canadian J. of Math.

[17] C. Laurent-Gengoux, F. Wagemann, Obstruction classes of crossed modules of Lie algebroid and Lie groupoids linked to existence of principal bundles, math.AT/0611226, submitted

[7] A. Fialowski, F. Wagemann, Cohomology and deformations of the infinite dimensional filiform Lie algebra $\mathfrak{m}_0$. math.RT/0703383, accepted in J. of Algebra

[25] K.-H. Neeb, F. Wagemann, Lie group structures on groups of smooth and holomorphic maps on non-compact manifolds. math.DG/0703460, submitted

[36] F. Wagemann, Deformations of Lie algebras arising from deformations of schemes, http://arxiv.org/abs/0707.4054, submitted

Introduction

The principal aim of my research is the theory of infinite dimensional Lie algebras from the homological point of view. The guiding philosophy when dealing with infinite dimensional objects is that a natural topology should come into play in order to tame the theory. For example, fix a topological Lie algebra $\mathfrak{g}$ and a given abelian topological Lie algebra $\mathfrak{m}$ and consider equivalence classes of *all* exact sequences

$$0 \to \mathfrak{m} \to \mathfrak{e} \to \mathfrak{g} \to 0. \tag{1}$$

By an exact sequence we mean here simply exactness as a sequence of discrete Lie algebras. From the point of view of topological Lie algebras, there are then non-trivial abelian extensions which are only non-trivial as extensions of topological vector spaces (in case $\mathfrak{g}$ and $\mathfrak{m}$ are infinite dimensional), there are topologically split abelian extensions of Lie algebras, and there are extensions which mix the two phenomena. In order to exclude the first type of abelian extensions and to concentrate on Lie algebraic phenomena, one restricts to topologically split exact sequences. This restriction is reflected at the level of Lie algebra cochains by taking *continuous cochains*. Indeed, fixing a splitting of (1), one can write $\mathfrak{e} = \mathfrak{g} \oplus \mathfrak{m}$ as topological vector spaces, and the bracket then becomes

$$[(x,a),(y,b)] \;=\; ([x,y], -x \cdot b + y \cdot a + \alpha(x,y)).$$

The continuity of the bracket and the section $\sigma : \mathfrak{g} \to \mathfrak{e}$ imply that $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is a *continuous* 2-cocycle on $\mathfrak{g}$ with values in $\mathfrak{m}$.

As illustrated in the above discussion, functional analysis enters in a rather algebraic way into our study. In fact, we are forced to work in a Fréchet space setting, as many infinite dimensional Lie algebras arise as spaces of sections of vector bundles on manifolds.

The infinite dimensional Lie algebras we are interested in arise as Lie algebras of vector fields on a manifold or as tensor products $A \otimes_{\mathbb{K}} \mathfrak{k}$ of a $\mathbb{K}$-Lie algebra $\mathfrak{k}$ with a commutative associative $\mathbb{K}$-algebra A, the tensor product being seen as a $\mathbb{K}$-Lie algebra. We call this latter type *current algebras*.

In the first section, I outline my research on the continuous cohomology of Lie algebras of vector fields, also called *Gelfand-Fuks cohomology*. Its difference with the algebraic or discrete cohomology theory is that cochains are supposed to be continuous with respect to a fixed topology on the Lie algebra and the module. I believe that although the subject has existed for more than thirty years and the fundamental question – meaning Bott's conjecture – was solved thirty yaers ago, many questions remain to be answered. Open questions about clear criteria for the collapse of the Gelfand-Fuks spectral sequences, the explicit computation of examples, the explicit formulae for cocycles, or the analoguous results for different types of cohomology like *Leibniz cohomology* support this belief. Furthermore, I think that the subject is not well covered by textbooks. For example, no textbook explains why the vanishing of the Pontryagin classes of a manifold makes it easier to compute Gelfand-Fuks cohomology, while this is well known to experts. Models, in the sense of rational homotopy, exist for Gelfand-Fuks cohomology, but there is no textbook which explains how to compute it explicitly, based on concrete examples as in [6] or references therein.

From this perspective, in the first section, I apply, roughly speaking, known methods and tools to different Lie algebras or different cohomology theories, in order to show the universality of the tools and to get new results – see sections 1.1, 1.2 and 1.4. Section 1.5 contains a discussion of the limits of Gelfand-Fuks cohomology for purely algebraic infinite dimensional Lie algebras. Indeed, each topology of the Lie algebra of derivations of the algebra of Laurent polynomials $\mathbb{K}[X, X^{-1}]$ seems artificial, but we do not know how to compute its *algebraic cohomology*, while its Gelfand-Fuks cohomology with respect to the subspace topology as the subalgebra of polynomial vector fields on the circle is well known.

The second section has a more homological algebraic flavour. We discuss the interpretation of the 3-cohomology of a Lie algebra as (equivalence classes of) *crossed modules*. A crossed module of Lie algebras is a morphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with a compatible action of $\mathfrak{n}$ on $\mathfrak{m}$. The point is that one easily constructs a crossed module associated to a given 3-cohomology classes, and that the construction gives some insight into the relation with other classes. The more traditional point of view is that crossed modules obstruct the existence of extensions. Geometry comes into play when one applies this algebraic setting to Lie groupoids and Lie algebroids. It is via these objects that the obstruction classes of Neeb in [22] are linked to gerbes on the manifold. Better understanding of the relation between crossed modules of Lie groupoids and gerbes is the next project in our collaboration with Karl-Hermann Neeb.

In the third section, we study the homological and Lie theoretical properties of holomorphic current algebras, i.e. of Lie algebras of holomorphic sections of trivial Lie algebra bundles on complex manifolds. More precisely, we determine their universal central extension in case the fiber Lie algebra is simple, we compute second continuous cohomology for general fiber Lie algebras, and we address the question of whether the topological groups of holomorphic maps into a complex Lie group carry a structure of an infinite dimensional Fréchet Lie group.

The last section treats deformations of infinite dimensional Lie algebras. We discuss first a link between deformations of Krichever-Novikov algebras and the moduli stack of curves. Our point of view is that the link can be easily understood by introducing a new stack, the *stack of deformations* of Lie algebras. The moduli stack is shown to have a natural morphism into this latter stack. We then study the properties of this morphism. It turns out to be almost a monomorphism, due to Pursell-Shanks theory which describes a manifold/variety by its Lie algebra of vector fields. Then we move on to the deformations of an infinite dimensional Lie algebra defined by generators and relations. The new interesting phenomenon we study is that although adjoint cohomology is infinite dimensional, true, i.e. unobstructed, deformations are only finitly generated.

For the convenience of the reader, we included glossaries on Gelfand-Fuks cohomology, on some selected subjects from homological algebra, and on infinite dimensional manifolds in three appendices.

1 Cohomology of Lie algebras of vector fields

This section treats *Gelfand-Fuks cohomology*, i.e. the cohomology of a topological Lie algebra of vector fields on a manifold M. We will be mainly interested in the Lie algebra $\mathrm{Vect}(M)$ of smooth vector fields on a smooth manifold M and in Lie algebras associated to a complex manifold X. In all cases, these Lie algebras will be topological Lie algebras with respect to the natural Fréchet[1] topology on the space of sections of a smooth (resp. complex) vector bundle over M (resp. X).

The Lie algebras showing up in the holomorphic context are the Lie algebra of holomorphic vector fields $\mathrm{Hol}(X)$ on X, the Lie algebra $\mathrm{Vect}_{1,0}(X)$ of type $(1,0)$ smooth vector fields (i.e. those smooth vector fields which are locally of the form

$$\sum_{i=1}^{n} f(z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)\frac{\partial}{\partial z_i}\quad)$$

and Lie algebras $\mathrm{Mer}_k(X)$ of meromorphic vector fields on X with possible pôles in k fixed points. The number of pôles of an element of $\mathrm{Mer}_k(X)$ may be strictly lower than k, or even zero. In case $X=\Sigma$ is a compact Riemann surface, the $\mathrm{Mer}_k(X)$ are also called *Krichever-Novikov* (Lie) *algebras.* The bracket of all these Lie algebras is inherited from the bracket of the Lie algebra $\mathrm{Vect}(X)$ of all smooth vector fields on X (resp. the Lie algebra $\mathrm{Vect}(X\setminus\{p_1,\ldots,p_k\})$ of smooth vector fields on $X\setminus\{p_1,\ldots,p_k\}$ for the fixed points $p_1,\ldots,p_k$ when we consider $\mathrm{Mer}_k(X)$).

Finally, the Lie algebra of formal vector fields W_n in n complex formal variables plays an important rôle. As a vector space, $W_n=\prod_{i=1}^n\mathbb{C}[[x_1,\ldots,x_n]]\frac{\partial}{\partial x_i}$. The topology on the space of formal series $\mathbb{C}[[x_1,\ldots,x_n]]$ is the simple convergence of coefficients which can also be described as the topology of the inverse limit where every copy of $\mathbb{C}^n$ is endowed with the usual topology. The bracket

[1] A real (or complex) Fréchet space is a complete, metrizable locally convex Hausdorff topological vector space. It is thus homeomorphic to a subspace of the countable infinite product of copies of $\mathbb{R}$ (or $\mathbb{C}$) endowed with the metric $d(x,y)=\sum_k\frac{2^{-k}|x_k-y_k|}{1+|x_k-y_k|}$.

is given by

$$\left[\sum_{i=1}^{n} f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i},\sum_{j=1}^{n} g_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}\right] = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f_i\frac{\partial g_j}{\partial x_i}-g_i\frac{\partial f_j}{\partial x_i}\right)\frac{\partial}{\partial x_j}.$$

Cohomology of a topological Lie algebra is computed as usual by a cochain complex. We tried to find the right setting for a derived functor approach to this topological cohomology, but we didn't succeed. Denote by $\mathfrak{g}$ one of the topological Lie algebras $\mathrm{Vect}(M)$, $\mathrm{Hol}(X)$, $\mathrm{Vect}_{1,0}(X)$, $\mathrm{Mer}_k(X)$ or W_n. $C^p(\mathfrak{g};\mathbb{C}) := \mathrm{Hom}_{\mathbb{C}-\mathrm{cont}}(\Lambda^p\mathfrak{g},\mathbb{C})$ the space of alternating continuous multilinear maps from $\mathfrak{g}$ to $\mathbb{C}$; it is the space of p-cochains in Gelfand-Fuks cohomology. The tensor products involved in $\Lambda^p\mathfrak{g}$ are always endowed with the topology of the *projective tensor product* $\otimes_\pi$ (with respect to the Fréchet topology on $\mathfrak{g}$). This is the topology characterized by the property that for any locally convex space E and any continuous bilinear map $\mathfrak{g}\times\mathfrak{g}\to E$, there is a unique continuous map $\mathfrak{g}\otimes_\pi\mathfrak{g}\to E$, see [30] §43. Nothing changes for the complex setting. The differential

$$dc(X_1,\ldots,X_{p+1}) = \sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{p+1}),$$

defined here for a p-cochain c, makes the collection of $C^p(\mathfrak{g};\mathbb{C})$s into a cochain complex, whose cohomology, denoted simply by $H^*(\mathfrak{g};\mathbb{C})$ is by definition the Gelfand-Fuks cohomology of $\mathfrak{g}$.

The task of computing $H^*(\mathrm{Vect}(M);\mathbb{C})$ for a given manifold M has been approached by several methods. Let us sketch two of them briefly in the following subsections. A third method is concerned with minimal models for the above defined cochain algebra in the spirit of rational homotopy theory, and is due to Haefliger. I gave a talk on this method in the Angers-Nantes working group on homotopy theory - it supported my belief that also Haefliger's approach can be formulated for Lie algebras of holomorphic vector fields.

1.1 The Gelfand-Fuks spectral sequence

The Gelfand-Fuks spectral sequence is a system of two spectral sequences, one nested in the other. It arises from two nested filtrations of the cochain complex $(C^q(\mathrm{Vect}(M);\mathbb{C}),d)$, whose elements are seen as generalized sections of some vector bundle. First one filters by the *support filtration*, and then for a fixed support, by the *order filtration*.

To be more precise, let us consider on M^q the vector bundle $\bigotimes^q TM := \bigotimes_{i=1}^q \mathrm{pr}_i^*(TM)$, where $\mathrm{pr}_i : M^q\to M$ is the projection onto the ith factor. It is clear how to antisymmetrize $\bigotimes^q TM$ in order to regard $c\in C^q(\mathrm{Vect}(M);\mathbb{C})$ as a *generalized section* of it, i.e. an element of the continuous dual of the space of sections. Now define M_k^q by

$$M_k^q := \{(x_1,\ldots,x_q)\in M^q \,|\, \forall\{i_1,\ldots,i_{k+1}\}\subset\{1,\ldots,q\} \,:\, \exists j,l \,:\, x_{i_j}=x_{i_l}\}.$$

This filtration of M^q gives rise to a filtration on $(C^p(\mathrm{Vect}(M);\mathbb{C}),d)$ by putting $c \in C^q_k(\mathrm{Vect}(M);\mathbb{C})$ if the section c is *concentrated on* M^q_k, i.e. $c(s)=0$ for all sections s with $\mathrm{supp}(s) \subset M^q \setminus M^q_k$. There is thus a finite filtration:

$$0 = C^q_0(\mathrm{Vect}(M);\mathbb{C}) \subset C^q_1(\mathrm{Vect}(M);\mathbb{C}) \subset \ldots \subset C^q_q(\mathrm{Vect}(M);\mathbb{C}) = C^q(\mathrm{Vect}(M);\mathbb{C}).$$

Observe that M^q_1 is the diagonal $\triangle \subset M^q$; this is why the filtration is sometimes called the *diagonal filtration*. $(C^q_1(\mathrm{Vect}(M);\mathbb{C}),d)$ is in fact a subcomplex, the *diagonal subcomplex* and denoted by $C^q_\triangle(\mathrm{Vect}(M);\mathbb{C})$. We denote by $H^*_\triangle(\mathrm{Vect}(M);\mathbb{C})$ the cohomology of the diagonal subcomplex.

The corresponding spectral sequence has its E_0-term

$$\tilde{E}_0^{k,*-k} \;=\; C^q_k(\mathrm{Vect}(M);\mathbb{C}) \,/\, C^q_{k-1}(\mathrm{Vect}(M);\mathbb{C}).$$

The RHS defines quotient complexes whose cohomology is computed by another spectral sequence, namely the one associated to the *order filtration*. Let us explain this filtration only for the diagonal subcomplex.

Here a generalized section $c \in C^q(\mathrm{Vect}(M);\mathbb{C})$ which is concentrated on $E \subset M^q$ is called *of order* $o(c) \leq l$ in case $c(s)=0$ for all sections s having trivial l-jet on E. On defines then

$$F^m C^q_1(\mathrm{Vect}(M);\mathbb{C}) \;=\; \{c \in C^q(\mathrm{Vect}(M);\mathbb{C}) \,|\, o(c) \leq q-m\}.$$

In the associated spectral sequence, one has:

Theorem 1 (Gelfand-Fuks) *Let M be an n-dimensional manifold and let W_n be the topological Lie algebra of n formal variables, endowed with the topology of simple convergence of the coefficients. In the Gelfand-Fuks spectral sequence for the diagonal cohomology, one has*

$$E_2^{p,q} \;=\; H_{-p}(M;\mathbb{C}) \otimes H^q(W_n;\mathbb{C}),$$

In this sense, the Gelfand-Fuks cohomology is a mixture of homology of the manifold and cohomology of the Lie algebra of formal vector fields.

In [33] we developed this kind of spectral sequence for the Lie algebra $\mathrm{Vect}_{1,0}(X)$. It works exactly in the same way, with the following modification: the holomorphic tangent bundle is a holomorphic bundle, thus it makes sense to take jets with respect to the holomorphic coordinates $z_1,\ldots,z_n$ for X of complex dimension n. The main result of [33] is the following theorem:

Theorem 2 *Let X be a complex manifold of complex dimension n and W_n as in theorem 1.*

There is a spectral sequence for the diagonal cohomology of $Vect_{1,0}(M)$ with second term

$$E_2^{p,q} \;\cong\; H_{\bar\partial}^{-p,0}(M)' \otimes H^q(W_n;\mathbb{C}).$$

where $(-)'$ denotes the dual vector space.

Due to the scarcity of non-zero cohomology spaces of W_1, this spectral sequence collapses in the case of a compact Riemann surface Σ, and permits to determine $H^2_\triangle(\mathrm{Vect}_{1,0}(\Sigma);\mathbb{C})$ which is thus of dimension g for Σ of genus $g>1$. Generators are explicitly described in [33].

Sketch of proof.

Denote simply by TX the holomorphic tangent bundle of X. The space $Vect_{1,0}(X)$ is the space of smooth sections of TX. $\bigotimes^q TX$ (see p. 6) is a holomorphic vector bundle on the complex manifold X^q. In particular, the notion of a trivial m-jet in z of a section of $\bigotimes^q TX$ (at a point $x\in X^q$) is independent on the choice of the local coordinate z. We restrict our setting now to the diagonal subcomplex $C^*_\triangle(Vect_{1,0}(X);\mathbb{C})$.

As before, the subspaces

$$F^m C^q_\triangle(Vect_{1,0}(X);\mathbb{C}) = \{c\in C^q_\triangle(Vect_{1,0}(X);\mathbb{C})\mid c \text{ has order } \leq q-m\}$$

induce a filtration on the diagonal complex. Indeed, $d(F^m C^q_\triangle(Vect_{1,0}(X);\mathbb{C}))\subset F^m C^{q+1}_\triangle(Vect_{1,0}(X);\mathbb{C})$, because the bracket in $Vect_{1,0}(X)$ involves only derivatives with respect to the z_i, $i=1,\ldots,n$. In addition, the filtration is exhaustive, because a section $s\in\Gamma(\bigotimes^q TX)$ with trivial ∞-jet in z is zero due to the z-dependence coming from the transition functions in the holomorphic bundle $\bigotimes^q TX$.

The term $E_0^{p,q}$ of the associated spectral sequence is the quotient of diagonal cochains $C^{p+q}_\triangle(Vect_{1,0}(X);\mathbb{C})$ which are of order $\leq q$ (i.e. vanishing on elements having trivial q-jets (in z)) factored by those of order $<q$.

Translate now elements of $E_0^{p,q}$ into generalized sections (suitably anti-symmetrized) of the bundle

$$\hat{\mathcal{E}}_0^{p,q} = \mathrm{Hom}(S^q \mathrm{norm}_{X^{p+q}}\triangle, (\bigotimes^{p+q} TX)|_\triangle).$$

Here, $\mathrm{norm}_{X^{p+q}}\triangle$ is the (holomorphic) normal bundle of the submanifold $\triangle(X)\subset X^{p+q}$ (it is the quotient bundle of the holomorphic tangent bundle of X^q by the holomorphic tangent bundle of $\triangle(X)$). The anti-symmetrized version is denoted

$$\mathcal{E}_0^{p,q} = \mathrm{Alt}(S^q \mathrm{norm}_{X^{p+q}}\triangle, (\bigotimes^{p+q} TX)|_\triangle).$$

Now, we pass to considerations on the fibers of these bundles: Let V denote the fibre of TX. The fibre of $\hat{\mathcal{E}}_0^{p,q}$ is thus

$$\mathrm{Hom}(S^q\left\{(V\oplus\ldots\oplus V)/V_\triangle\right\}, \underbrace{V\otimes\ldots\otimes V}_{p+q}).$$

Here, $V_\triangle$ stands for the image of the diagonal inclusion $V\to V\oplus\ldots\oplus V$. Because of the restriction of the order filtration of Gelfand-Fuks to jets involving just z, only the fibre of the holomorphic tangent space is showing up in the above formula. By a Koszul resolution, the term $S^q\left\{(V\oplus\ldots\oplus V)/V_\triangle\right\}$ is translated into factors $S^{q-i+1}\left(\bigoplus_{p+q} V\right) \otimes \Lambda^{i-1} V$. The alternating maps from the factor $S^*\left(\bigoplus_{p+q} V\right)$ to $\bigotimes^* V$ give the Gelfand-Fuks complex for W_n, the factor $\Lambda^* V$ will give differential forms. In fact, W_n is here seen as the Lie algebra of formal variables on the dual V', instead of V; as $V' \cong V$, one then attributes the dual rather to the space of forms then to W_n, cf details in [10] pp. 144–147. We get:

$$E_1^{p,q} = \Omega^{-p,0}(M)' \otimes H^q(W_n; \mathbb{C}).$$

The differential $d_1^{p,q}$ identifies with ∂ acting on $\Omega^{-p,0}(X)$, thus:

$$d_1^{p,q} = \partial \otimes id : \Omega^{-p,0}(X)' \otimes H^q(W_n; \mathbb{C}) \to \Omega^{-(p+1),0}(X)' \otimes H^q(W_n; \mathbb{C}).$$

This shows the theorem. □

An open question here is whether the overall Gelfand-Fuks sequence collapses, too.

1.2 Čech- or simplicial methods

Bott-Segal [2] and Haefliger [15] showed in the mid 1970ies that Gelfand-Fuks cohomology $H^*(\mathrm{Vect}(M); \mathbb{C})$ is isomorphic to the singular cohomology $H^*(\Gamma(E); \mathbb{C})$ of the space of sections $\Gamma(E)$ of some fiber bundle E on M, associated to the tangent bundle TM with typical fiber X_n. The space X_n is such that it has as its singular cohomology the Gelfand-Fuks cohomology of $\mathrm{Vect}(\mathbb{R}^n)$, if n is the dimension of M. It expresses the idea that $H^*(\mathrm{Vect}(M); \mathbb{C})$ may be computed by patching together the local results, i.e. the result for $\mathrm{Vect}(\mathbb{R}^n)$.

Later, Kawazumi [17] showed a similar theorem for the Lie algebra of holomorphic vector fields $Hol(\Sigma)$ on an open Riemann surface Σ. In my thesis, whose main result is published in [32], the reasoning of Kawazumi (who uses Bott-Segal's method) has been adapted to the n-dimensional case. It is based on the following three steps (which are only those differing from previous works – we cannot be exhaustive here !); let X be a complex Stein manifold:

(a) First step: the *punctual* and the *local* case.

   (i) $X = \mathbb{C}^n$ and $\mathrm{Hol}(X) = W_n$, the formal version of the Lie algebra of holomorphic vector fields. There is a space X_n such that its singular cohomology is the Gelfand-Fuks cohomology of W_n.

   (ii) $X = \mathbb{C}^n$ and $\mathrm{Hol}(X) = \mathrm{Hol}(\mathbb{C}^n)$, the Lie algebra of holomorphic vector fields on $\mathbb{C}^n$. The singular cohomology of X_n is isomorphic to the Gelfand-Fuks cohomology of $\mathrm{Hol}(\mathbb{C}^n)$.

   (iii) $X = U$ and $\mathrm{Hol}(X) = \mathrm{Hol}(U)$ for a contractible Stein open set U. The singular cohomology of X_n is still isomorphic to the Gelfand-Fuks cohomology of $\mathrm{Hol}(U)$.

(b) Second step: adapt the first step to pass to a good Stein covering of contractible sets $\mathcal{U} = \{U_i\}_{i \in I}$ of X. This means that all finite intersections are still Stein and contractible. We denote for a finite set of indices σ by $U_\sigma = \bigcap_{i \in \sigma} U_i$ and $U^\sigma = \bigcup_{i \in \sigma} U_i$. We assume that all U_i, U_σ and U^σ are non-empty. First we show that there is a *fundamental map*

$$\hat{f}_\sigma \,:\, C^*(Hol(U_\sigma)) \,\to\, \Omega^*(U^\sigma; C^*(Hol(\mathbb{C}^n)))$$

which links the cohomology of the cosimplicial Lie algebra $Hol_{\mathcal{U}}(X)$ associated to (the thickened nerve of) the covering with the $C^*(Hol(\mathbb{C}^n))$-valued differential forms on the cosimplicial manifold associated to (the thickened nerve of) the covering. Secondly, we pass by cohomology equivalence to the subcomplex of *holomorphic* differential forms thanks to the Stein hypothesis.

(c) The last step is the *addition theorem* of Kawazumi which identifies the cohomology of the cosimplicial Lie algebra with that of $\mathrm{Hol}(X)$ in case X is itself a Stein manifold. It follows from Bott-Segal's work (cf cor. 5.8 of [2]) that the spaces in the range of the fundamental map fit together to a cosimplicial model for the space of sections of the bundle E.

Carrying out these steps, we get:

Proposition 1

$$H^*(Hol(\mathbb{C}^n); \mathbb{C}) \cong H^*(W_n; \mathbb{C}) \cong H^*(X_n; \mathbb{C})$$

with the same space X_n as in Bott-Segal's work, which is in fact a complex manifold.

Sketch of proof. The continuous dual of the space of formal series W_n in the topology coming from the simple convergence of coefficients can be identified with a space of polynomials, cf [30] thm. 22.1, p. 228. The continuous dual of the space $Hol(\mathbb{C}^n)$ of holomorphic vector fields or functions on $\mathbb{C}^n$ can be identified with a space of entire functions of exponential type, cf [30] thm. 22.2, p. 233. As always in Gelfand-Fuks theory, we regard these continuous duals as vector spaces, without fixing a topology. The continuous cochain complexes are then suitably antisymmetrized completed tensor products of these duals.

The Taylor expansion map

$$Hol(\mathbb{C}^n) \,\to\, W_n, \;\; \sum_{i=1}^n f_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i} \,\mapsto\, \sum_{i=1}^n \mathrm{jet}_x^\infty(f_i)(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}$$

associates to a field its infinite Taylor jet at a point $0 \in \mathbb{C}^n$. It is a continuous injective morphism of Lie algebras with dense image, inducing an injective restriction map

$$\phi : C^*(W_n; \mathbb{C}) \,\to\, C^*(Hol(\mathbb{C}^n); \mathbb{C}).$$

The Taylor expansion map and ϕ are equivariant with respect to the action of the Euler vector field $e_0 := \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$. e_0 acts as a grading element on both Lie algebras. Theorem 1.5.2 p. 45 [10] implies that the computation of the Gelfand-Fuks cohomology of both Lie algebras may be reduced to the subcomplex of e_0-invariant cochains, i.e. cochains of total degree 0. But the subcomplexes $C^*(W_n;\mathbb{C})^{e_0}$ and $C^*(Hol(\mathbb{C}^n);\mathbb{C})^{e_0}$ coincide under ϕ. Therefore, $Hol(\mathbb{C}^n)$ and W_n have isomorphic cohomology.

The manifold X_n is the inverse image of an open neighborhood of the $2n$-skeleton of the base space of the universal $GL_n(\mathbb{C})$-bundle in the total space, cf [2] p. 290. Its cohomology is best described by saying that it is a quotient of the *Weil algebra* $\Lambda\, gl_n(\mathbb{C})^* \otimes S\, gl_n(\mathbb{C})^*$. $\square$

Here we adapted Kawazumi's proof; it is also possible to adapt Bott-Segal's proof. We choose to omit the other steps in the proof here.

Summerizing:

Theorem 3 *Let X be a complex Stein manifold of complex dimension n. Then there exists a fiber bundle E associated to the holomorphic tangent bundle of X and with typical fiber X_n such that*

$$H^*(Hol(X);\mathbb{C}) \cong H^*(\Gamma E;\mathbb{C}).$$

One can extend this theorem formally to non-Stein manifolds X by defining the holomorphic Gelfand-Fuks cohomology of X as the cohomology of the cosimplicial Lie algebra $Hol_{\mathcal{U}}(X)$ with respect to a good Stein covering of X (whose existence is then a condition on X).

We did explicit computations once again for (compact) Riemann surfaces.

1.3 Cohomology of Krichever-Novikov algebras

Inspired by Millionshchikov's spectral sequence [21] derived from the almost grading of Krichever-Novikov Lie algebras, we became interested in the cohomology of Krichever-Novikov algebras.

Observe first that a dense subalgebra leads to isomorphic Gelfand-Fuks cohomology:

Lemma 1 *If $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is dense in the subspace topology, then we have an isomorphism of graded vector spaces*

$$C^*(\mathfrak{g};\mathbb{C}) \to C^*(\mathfrak{h};\mathbb{C}),$$

induced by i, and consequently an isomorphism of their Gelfand-Fuks cohomologies.

In [31], we observed that the problem of the computation of the cohomology of Krichever-Novikov algebras is not well posed, as there is no natural topology on Krichever-Novikov algebras. The main result of [31] is the quite simple fact that if $Mer_k(\Sigma)$ carries the topology induced from the embedding $Mer_k(\Sigma) \subset Hol(\Sigma \setminus \{p_1, \ldots, p_k\})$ (where $p_1, \ldots, p_k$ are the fixed points where fields are allowed to have pôles), then the two topological Lie algebras have the same Gelfand-Fuks cohomology, as one is dense in the other. The Gelfand-Fuks cohomology of $Hol(\Sigma \setminus \{p_1, \ldots, p_k\})$ can be deduced from Kawazumi's result [17].

The density $Mer_k(\Sigma) \subset Hol(\Sigma \setminus \{p_1, \ldots, p_k\})$ follows from a theorem of Behnke-Stein [1]. It is the prototype of density theorems, a general version of which can be stated in the following situation:

Consider a connected compact complex manifold X and $Y \subset X$ a complex codimension 1 submanifold which is given by an effective divisor D_Y, which we suppose to be ample. X may then be embedded into projective space, and Y is the affine subvariety given by the complement of a hyperplane.

Theorem 4 *Let E denote an algebraic vector bundle on the affine algebraic variety $X \setminus Y$. Assume that there is an algebraic vector bundle F on projective space such that E is a subbundle or a quotient of the restriction of F to X.*

Then, $E(X \setminus Y) \subset E^{\mathrm{an}}(X \setminus Y)$ is dense in the subspace topology, where $E(X \setminus Y)$ is the space of regular sections on $X \setminus Y$ and $E^{\mathrm{an}}(X \setminus Y)$ is the space of holomorphic sections on $X \setminus Y$.

This theorem applies to any kind of algebras which are spaces of sections of a vector bundle, and shows the density of the subalgebra of algebraic sections (with the induced topology) in the space of holomorphic sections. For associative algebras, one gets isomorphism theorems on continuous cyclic or Hochschild cohomologies.

In [35], we determine representative cocycles for a set of generators of the Gelfand-Fuks cohomology spaces $H^2(Hol(\Sigma), \mathcal{F}_\lambda(\Sigma))$ in terms of geometric objects. Here $\mathcal{F}_\lambda(\Sigma)$ is the space of holomorphic λ-densities on the open Riemann surface Σ, i.e. for $\lambda \in \mathbb{C}$, $\mathcal{F}_\lambda(\Sigma)$ is a space of holomorphic functions $g(z)(dz)^\lambda$ on Σ on which a vector field $X = f(z)\frac{\partial}{\partial z}$ acts by

$$f(z)\frac{\partial}{\partial z} \cdot g(z)(dz)^\lambda \,=\, (fg'(z) + \lambda f' g(z))(dz)^\lambda.$$

Kawazumi computed in *loc. cit.* the dimensions of the cohomology spaces and the principal symbol of the cocycles representing the cohomology classes. We used these results to construct cocycles in terms of affine and projective connections T and R on Σ. An *affine connection* T and a *projective connection* R are geometric objects on Σ which are characterized by their transformation behaviour. More precisely, under a coordinate change $z_\beta = h(z_\alpha)$ on an intersection of coordinate domains $U_\alpha \cap U_\beta$, R is supposed to transform according to

$$R_\beta\,(h')^2 \,=\, R_\alpha \,+\, S,$$

where the *Schwarzian derivative* S is given by

$$S \,=\, \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2.$$

In the same way, T is supposed to transform according to

$$T_\beta \, h' = T_\alpha + \frac{h''}{h'}.$$

The existence of this kind of connections expresses in fact a geometric structure on the manifold in question by choosing a *projective atlas* (resp. an *affine atlas*) in which the R_α (resp. the T_α) are zero. Affine connections (and thus projective connections) exist on any open Riemann surface. In the case of compact Riemann surfaces, they exist only for genus 1.

Now, let me explain on an example how to construct cocycles by examining the transformation behaviour of these principal symbols, and then compensating additional terms by adding differential polynomials in T and R: the central extensions of $Mer_k\Sigma$ (and of $Hol(\Sigma \setminus \{p_1, \ldots, p_k\})$ where here Σ is a compact Riemann surface) are given by

$$c(f,g) = \frac{1}{2}\int_\Sigma \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix}.$$

But when one computes the transformation behaviour under coordinate change of the determinant (using that f and g are coefficient functions of vector fields), one finds that the determinant does not behave as a 2-form. Actually, it does behave as a holomorphic 1-form, up to additional terms. The up shot of this discussion is that

$$c(f,g) = \int_\Sigma \left(\frac{1}{2} \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} - 2R \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \right) dz \wedge \bar{\omega}$$

is the right expression; here $\omega \in H^1(\Sigma)$ is a generator, and we had to add a determinant which is a coboundary and thus does not change the cohomology class of the cocycle, but which is necessary in order to make the result a globally defined 1-form.

We refer to [35] for the other explicit formulae. Let us remark that our method of determining cocycles for $\mathrm{Hol}(\Sigma)$ from cocycles for $\mathrm{Vect}(S^1)$ using affine and projective connections was used afterwards by Bouarroudj and Gargoubi in [3].

1.4 Leibniz cohomology of vector fields

The notion of a *Leibniz algebra* is a generalization of the notion of Lie algebra: the bracket is only supposed to satisfy a Jacobi identity, but is not necessarily antisymmetric. Leibniz algebras are algebras over a certain operad. Cohomology and deformation theory follow easily from the corresponding picture for general quadratic operads. As Lie algebras are Leibniz algebras, it makes sense to apply the Leibniz cohomology theory to Lie algebras, and to compute in this way new invariants (i.e. cohomology classes) for Lie algebras.

In [9], Alessandra Frabetti and I adapted the Gelfand-Fuks spectral sequences to the Leibniz setting. This is rather easy, as the only change is that the cochain complex is now consisting of all cochains, instead of consisting only of antisymmetric cochains. Then, using the cohomology computations of Lodder [19] for the Leibniz cohomology (in the Gelfand-Fuks sense) $HL^*(W_1;\mathbb{C})$ of W_1, we deduced the diagonal Leibniz cohomology $HL^*_\triangle(\mathrm{Vect}(S^1);\mathbb{C})$ of the Lie algebra of smooth vector fields on the circle.

The Leibniz cohomology of W_1 is the dual Leibniz algebra generated by the Godbillon-Vey cocycle θ_0 and a new generator β in degree 4. The cup products θ_0^2 and $\beta\cup\theta_0$ are zero. The cup product $\beta^2\cup\beta$ is equal to $2\beta\cup\beta^2$, because of the relation of dual Leibniz algebra and the fact that β has even degree. Hence the only higher order local[2] Leibniz cocycles are $\beta^k := \beta\cup\beta^{k-1}$ and $\theta_0\cup\beta^k$.

The main theorem of [9] reads:

Theorem 5 *The diagonal Leibniz cohomology of* $\mathrm{Vect}\, S^1$ *is the graded vector space spanned by the classes of the local cocycles*

$$\begin{array}{ll} \theta_0\cup\beta^r & \textit{in degree } 3+4r, \textit{ for } r\geq 0 \\ \beta^s & \textit{in degree } 4s, \textit{ for } s\geq 1 \end{array}$$

and by the classes of the diagonal cocycles

$$\begin{array}{ll} \omega_r & \textit{in degree } 2+4r, \textit{ for } r\geq 0 \\ \gamma_s & \textit{in degree } 4s-1, \textit{ for } s\geq 1 \end{array}$$

where $\omega_0=\omega$ *is the Gelfand-Fuks cocycle in degree* 2 *and* ω_r, γ_s *determine new invariants for* $r,s\geq 1$.

The first new class γ_1 can be represented by the cocycle

$$\gamma_1(l,f,g) \;=\; \int_{S^1} l'(t)\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}(t)\; dt$$

for three vector fields f,g,l on S^1 (given by their coefficient functions). It can be obtained from β by integration over the fiber. The higher order classes are obtained in the same way from the products β^r.

More informations on these cocycles may be found in appendix A. It remains to understand whether in the Leibniz setting the overall Gelfand-Fuks cohomology is also just multiplicatively generated from the diagonal cohomology. This would permit to compute all Leibniz cohomology of $\mathrm{Vect}(S^1)$.

1.5 Skryabin's approach to cohomology of vector fields

Observe that the computations of algebraic or discrete cohomology (in opposition to Gelfand-Fuks cohomology) of infinite dimensional Lie algebras are usually very hard. For example, Dimitry Millionshchikov and I, we do not know any computation of the *algebraic* cohomology of the Lie algebra

$$\mathrm{Vect}_{\mathrm{pol}}(S^1) \;=\; \bigoplus_{i\in\mathbb{Z}}\mathbb{C}x^{i+1}\frac{d}{dx}$$

[2] Local means here support preserving, see appendix A.

of polynomial vector fields on the circle (we do know low degree terms), whereas the Gelfand-Fuks cohomology of $\mathrm{Vect}_{\mathrm{pol}}(S^1)$ is well-known (using lemma 1) by density of $\mathrm{Vect}_{\mathrm{pol}}(S^1)$ in $\mathrm{Vect}(S^1)$ in the subspace topology. For example, the passage to the subcomplex of Euler-invariant cochains as in the reasoning of proposition 1 does not reduce the computation to finite dimensional cochain spaces for $\mathrm{Vect}_{\mathrm{pol}}(S^1)$.

There are approaches to computations of algebraic cohomology of infinite dimensional Lie algebras. I have already mentioned Millionshchikov's approach using the almost grading of Krichever-Novikov algebras. Unfortunately, it does not provide a way to compute the algebraic cohomology of $\mathrm{Vect}_{\mathrm{pol}}(S^1)$ – it rather uses it.

Skryabin lays out in [29] an approach to algebraic cohomology computations via the theory of representations. Unfortunately, for the moment, it works only in degree 1, and to a limited extend in degree 2. On the other hand, it is completely general and works over a commutative associative unital ring R such that 2 is invertible in R (for the result which is of interest to us, also 3 must be invertible).

Indeed, let W be a projective R-module of finite rank $n > 0$ which is a submodule of the module of derivations $\mathrm{Der}(R)$. Define Ω^1 to be $\mathrm{Hom}_R(W, R)$ and suppose $\Omega^1 = R\,dR$ where for all $f \in R$, df is defined by $df(D) = Df$ for all $D \in W$. Let $\mathfrak{g} = \Omega^1 \otimes_R W$ be the Lie algebra of W-valued differential forms. Skryabin defines then a category $\mathcal{C}_1$ of simultaneous R-, W- and $\mathfrak{g}$-modules with compatibility relations between these actions. He analyzes in detail cocycles on W with values some object of $\mathcal{C}_1$ by their order, and gives conditions when they are differential operators. As a result, Skryabin is able to compute all 1-cohomology with values in an object of $\mathcal{C}_1$. Finally, using the trick $H^2(W, V) \hookrightarrow H^1(W, \mathrm{Hom}_{\mathbb{Z}}(W, V))$ (true for *central* extensions, meaning a *trivial* W-action on V), he can use his previous results to to obtain the universal $\mathbb{Z}$-split central extension of W. The result ([29] thm. 7.1, p. 103) is that if $n > 1$, every $\mathbb{Z}$-split central extension splits, but if $n = 1$, the universal central extension has center $H^1(\Omega)$, cohomology in the de Rham complex $\Omega^* = \Lambda\Omega^1$. This is the expected result from all computations in continuous cohomology, but here it holds in *algebraic* cohomology.

For $\mathrm{Vect}_{\mathrm{pol}}(S^1)$, this does not give anything new, as we know how to compute H^2 algebraically. But for example for $Mer_k(\Sigma)$, this is new: $Mer_k(\Sigma)$ is a projective $\mathcal{O}_{\Sigma_k}$-module as a space of sections on the affine smooth algebraic variety $\Sigma_k = \Sigma \setminus \{p_1, \ldots, p_k\}$, and of rank 1. It suffices to compute $H^1(\Omega)$. But this is just the de Rham cohomology, as it coincides with the cohomology computed by the complex of algebraic forms on a smooth affine variety. In conclusion, $Mer_k(\Sigma)$ has a universal $\mathbb{Z}$-split central extension given by a center of dimension $b_1(\Sigma_k)$, the first Betti number of Σ_k. This is known as Feigin's conjecture, cf [21].

Note that the setting of the article of Skryabin is a special case of the setting of *Lie-Rinehart algebras* of Rinehart [27], Huebschmann [16] and similar to the one of *modular Lie algebras* of Grabowski [12], Siebert [28]. Observe further that Lie-Rinehart algebras appear as modules of sections of Lie algebroids.

2 Crossed modules and 3-cohomology

2.1 Constructing crossed modules

A *crossed module* is a morphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with a compatible action of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations. In the same way as 2-cohomology classifies central extensions, 3-cohomology classifies crossed modules, i.e. a 3-cohomology class corresponds uniquely to an equivalence class of a certain crossed module. See appendix B for more precisions on crossed modules and their relation to 3-cohomology.

My interest in crossed modules of Lie algebras goes back to a question of Jean-Louis Loday about a crossed module representing the Godbillon-Vey cocycle θ_0 whose cohomology class $[\theta_0] \in H^3(W_1;\mathbb{C}) \cong H^3(Vect(S^1);\mathbb{C})$ is a generator. The cocycle is given by the value at $t = 0$ of the function

$$\theta_t(f,g,h) = \begin{vmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{vmatrix},$$

where f, g, h are coefficient functions of three vector fields.

I rediscovered (after Gerstenhaber 40 years ago) that crossed modules of Lie algebras for a Lie algebra $\mathfrak{g}$ may be constructed by splicing together a short exact sequence of $\mathfrak{g}$-modules

$$0 \to V' \to V \to V'' \to 0$$

with an abelian extension given by a 2-cocycle α

$$0 \to V'' \to V'' \times_\alpha \mathfrak{g} \to \mathfrak{g} \to 0.$$

The cohomology class to which the resulting crossed module corresponds is just $[\partial\alpha]$, the image of α under the connecting homomorphism corresponding to the above short exact sequence of $\mathfrak{g}$-modules.

In any category of $\mathfrak{g}$-modules which has enough injectives, or more precisely, in which any module may be embedded in one having trivial 3-cohomology, one has a representative of this kind for each crossed module.

Now, the key step (cf appendix A) in order to construct in this way the crossed module corresponding to $[\theta_0]$ is that

$$d^{\mathbb{C}}\alpha = d_{\mathrm{dR}}\theta_t, \tag{2}$$

because it is this equation which implies (by a straight forward computation) that the image of α under the connecting homomorphism (corresponding to the de Rham short exact sequence of coefficients mentioned below) is θ_0. Here $d^{\mathbb{C}}$ is the Lie algebra coboundary with values in the trivial module $\mathbb{C}$, d_{dR} is the de Rham differential of the function θ_t, and α is the following 2-cocycle of W_1 with values in formal 1-forms Ω^1:

$$\alpha(f,g) = \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}.$$

The short exact sequence of W_1-modules is the de Rham sequence

$$0 \to \mathbb{C} \to \Omega^0 \to \Omega^1 \to 0.$$

The construction of the crossed module

$$0 \to \mathbb{C} \to \Omega^0 \to \Omega^1 \times_\alpha W_1 \to W_1 \to 0$$

is the main result of [34].

[36] uses these ideas in order to construct a crossed module corresponding to the cocycle $\langle [,], \rangle$ whose class generates $H^3(\mathfrak{g};\mathbb{C})$ for a simple finite dimensional Lie algebra $\mathfrak{g}$. Here $\langle , \rangle$ is the Killing form on $\mathfrak{g}$. The origin of this work is the observation that the above crossed module for θ_0 on W_1 restricts to one on $sl_2(\mathbb{C}) \subset W_1$ representing a multiple of $\langle [,], \rangle$. The above de Rham sequence becomes for a general simple Lie algebra the restricted[3] dual sequence

$$0 \to L_0^\sharp \to M_0^\sharp \to N_0^\sharp \to 0$$

of the sequence

$$0 \to N_\lambda \to M_\lambda \to L_\lambda \to 0$$

which defines the irreducible quotient L_λ of the Verma module M_λ of highest weight λ, taken in the special case $\lambda = 0$ (with $L_0 \cong \mathbb{C}$).

The proof that the so constructed crossed module

$$0 \to \mathbb{C} \to M_0^\sharp \to N_0^\sharp \times_\alpha \mathfrak{g} \to \mathfrak{g} \to 0$$

for a certain α gives a multiple of the class of $\langle [,], \rangle$ uses spectral sequence techniques in order to show that the induced map

$$H^3(\mathfrak{g}, L_0^\sharp) \to H^3(\mathfrak{g}, M_0^\sharp)$$

is the zero map.

2.2 Lie groupoids and gerbes

While working on [36], I communicated with Karl-Hermann Neeb who also worked on crossed modules at that time. We had different points of view on the subject. I thought the *constructive* point of view is more interesting. He was more interested in the *obstructive* point of view, i.e. given a morphism $\psi : \mathfrak{g} \to \mathfrak{out}(\mathfrak{a})$ of a Lie algebra $\mathfrak{g}$ into the exterior derivations of a Lie algebra $\mathfrak{a}$, is there an extension of $\mathfrak{g}$ by $\mathfrak{a}$ where ψ comes from ? The answer to this question lies in a crossed module associated to ψ and whose class is the obstruction to the existence of the extension. Neeb formulated carefully this theory in the topological framework [22]. He applied it to construct a crossed module of topological Lie algebras associated to a K-principal bundle P on a manifold

[3] The restricted dual $M^\sharp$ of a graded module $M = \bigoplus_i M_i$ is $M^\sharp = \bigoplus_i M_i^*$, the direct sum of the duals of the graded pieces.

M and a central extension $\hat{K}$ of the structure group by some abelian group Z. The obstruction class of the crossed module can then be identified with a class in $H^3_{\mathrm{dR}}(M)$, the third de Rham cohomology of M. It remained the question whether this class was the full obstruction to the existence of a $\hat{K}$-principal bundle $\hat{P}$ on M with $\hat{P}/Z \cong P$, and the question on the link to gerbes on M, objects that are also classified by 3-cohomology classes on M.

In [18], Camille Laurent-Gengoux and I solved these questions by showing that under the set up of topological Lie algebras lies a set up of Lie algebroids (the Atiyah algebroid) and of Lie groupoids, in which the questions are already solved by Mackenzie. The main result of [18] is the identification of the different obstruction classes in order to link the two frameworks:

Theorem 6 *Given a K-principal bundle P on a manifold M and a central extension*

$$1 \to Z \to \hat{K} \to K \to 1$$

of the structure group, the 3-cohomology class $[\omega_{\text{top alg}}]$ of the crossed module of topological Lie algebras

$$0 \to \mathfrak{c} \to \hat{\mathfrak{n}} \to \mathfrak{aut}(P) \to \mathrm{Vect}(M) \to 0$$

defines the same de Rham cohomology class as the 3-cocycle ω_{grp} associated to the crossed module of Lie groupoids

$$0 \to M \times Z \to P_K(\hat{K}) \to (P \times P)/K \to M \times M \to 0.$$

Here $\mathfrak{aut}(P)$ is the topological Lie algebra of K-invariant vector fields on P, $\mathfrak{n}$ is the subalgebra of vertical vector fields, the *gauge* Lie algebra, and $\mathfrak{c} = \mathcal{C}^\infty(M, \mathfrak{z})$ where $\mathfrak{z}$ is the Lie algebra of Z. On the other hand, $P_K(\hat{K})$ is the Lie groupoid given by the K-equivariant maps from the fibers P_m to $\hat{K}$, and $(P \times P)/K$ is the Lie groupoid of the Atiyah algebroid.

The proof uses Deligne cohomology. As a consequence of the theorem, one sees that *up to torsion* Neeb's class is the full obstruction of the existence problem.

Another result of [18] is the bijection between crossed modules of Lie groupoids a certain kind on M and gerbes with abelian, trivialized band which induces the identity on the obstruction classes:

Theorem 7 *There is a one-to-one correspondence between gerbes with abelian, trivialized band on a connected manifold M, and crossed modules of Lie groupoids with trivial kernel and cokernel $M \times M$, which induces a one-to-one correspondence between cohomology classes when passing to equivalence classes.*

The deeper reason for this theorem is the fact that all Lie groupoids satisfying the conditions of the theorem come from an extension problem of the structure group in a principal bundle (for more precision, see prop. 1 in *loc. cit.*).

3 Holomorphic current algebras

A second important class of infinite dimensional Lie algebras is the class of *current algebras*, i.e. of Lie algebras of the form $A \otimes \mathfrak{k}$ where A is a commutative unital associative algebra and $\mathfrak{k}$ a Lie algebra, the bracket on $A \otimes \mathfrak{k}$ being:

$$[a \otimes x, b \otimes y] \;=\; ab \otimes [x, y],$$

for $a, b \in A$ and $x, y \in \mathfrak{k}$. We will always reserve $\mathfrak{k}$ for the Lie algebra tensor factor, and $\mathfrak{g}$ for the whole Lie algebra $\mathfrak{g} = A \otimes \mathfrak{k}$. Current algebras play a rôle in K-theory as $sl_n(A)$ or $gl_n(A)$ or their direct limits, in conformal field theory where fields are $\mathfrak{k}$-currents on a manifold, or in deformation theory where base change leads automatically to tensor products $A \otimes \mathfrak{k}$. For $A = \mathcal{C}^\infty(M)$ for a manifold M, $A \otimes \mathfrak{k}$ can be identified with the Lie algebra $\mathcal{C}^\infty(M, \mathfrak{k})$ of $\mathfrak{k}$-valued functions on M. Here the tensor product must be interpreted as the topological π-tensor product in case $\mathfrak{k}$ is topological.

My interest in the subject lies in *holomorphic current algebras*, i.e. current algebras of the form $\mathcal{O}(X, \mathfrak{k})$ for a complex (Stein) manifold X and some Lie algebra $\mathfrak{k}$, often taken to be finite dimensional.

3.1 Universal central extension

A first paper on the subject concerns the universal central extension of $\mathcal{O}(X, \mathfrak{k})$ for a finite dimensional (semi)simple complex Lie algebra $\mathfrak{k}$. The identification of the universal central extension of the Lie algebras $A \otimes \mathfrak{k}$ has a long history. Let us cut it short by stating that Maier addressed in [20] the question for a Fréchet algebra A, and solved it using the notion of the universal module of differentials $\Omega^1(A)$, which is a Fréchet A-module. The center of the universal central extension is then given by $\Omega^1(A) \,/\, \overline{dA}$.

In [24], Neeb and I identified the Fréchet module $\Omega^1(A)$ for the algebra $A = \mathcal{O}(X)$ of holomorphic functions on a Stein manifold with the Fréchet A-module of holomorphic 1-forms on X, thereby solving the question of the universal central extension in this case. More precisely, the de Rham differential

$$d : \mathcal{O}(X) \to \Omega^1(X)$$

is a continuous derivation of $\mathcal{O}(X)$-modules, and gives by universality of the pair $(\Omega^1(\mathcal{O}(X)), d_{\mathcal{O}(X)})$ rise to a unique morphism of $\mathcal{O}(X)$-modules

$$\gamma_X : \Omega^1(\mathcal{O}(X)) \to \Omega^1(X) \quad \text{with} \quad \gamma_X \circ d_{\mathcal{O}(X)} = d.$$

Theorem 8 *Let X be a Stein manifold. Then the map*

$$\gamma_X : \Omega^1(\mathcal{O}(X)) \to \Omega^1(X)$$

is an isomorphism of topological $\mathcal{O}(X)$-modules.

The theorem can be shown directly in the case of an open set $U \subset \mathbb{C}^n$ such that the restrictions of polynomials are dense in U. The general case proceeds by sheafifying the map γ.

The theorem is true more generally on a *Riemannian domain* X over a Stein manifold Y, i.e. for a complex manifold X together with a holomorphic map $p : X \to Y$ which is everwhere regular.

3.2 Second cohomology of current algebras

In a second article [25] together with Karl-Hermann Neeb, we computed the second cohomology of arbitrary current algebras, and this also in the topological setting (where, for example, A is a Fréchet algebra and $\mathfrak{k}$ finite dimensional, not necessarily semisimple, and the cohomology is computed via continuous cochains).

In order not to enter the long history on cohomology computations for current algebras, let me cite only papers of Haddi [14] and Zusmanovich [38], where the corresponding homology is computed in an algebraic framework and with non-natural splittings of quotient maps. Our contribution is to get rid of the splittings, and to have a framework suitable for continuous cohomology.

Our point of view is the following: we consider $\mathfrak{z}$-valued 2-cochains on $\mathfrak{g}$ as linear functions $f : \Lambda^2(\mathfrak{g}) \to \mathfrak{z}$. Such a function is a 2-*cocycle* if and only if it vanishes on the subspace $B_2(\mathfrak{g}) := \mathrm{im}(\partial)$ of 2-boundaries, which is the image of the linear map

$$\partial : \Lambda^3(\mathfrak{g}) \to \Lambda^2(\mathfrak{g}), \quad x \wedge y \wedge z \mapsto [x,y] \wedge z + [y,z] \wedge x + [z,x] \wedge y.$$

In view of the Jacobi identity, $B_2(\mathfrak{g})$ is contained in the subspace $Z_2(\mathfrak{g})$ of 2-cycles, which is the kernel of the linear map $b_{\mathfrak{g}} : \Lambda^2(\mathfrak{g}) \to \mathfrak{g}, x \wedge y \mapsto [x,y]$. The quotient space

$$H_2(\mathfrak{g}) := Z_2(\mathfrak{g})/B_2(\mathfrak{g})$$

is the second homology space of $\mathfrak{g}$.

A 2-cocycle f is a *coboundary* if it is of the form $f(x,y) = d_{\mathfrak{g}}\ell(x,y) := -\ell([x,y])$ for some linear map $\ell : \mathfrak{g} \to \mathfrak{z}$. We write $B^2(\mathfrak{g},\mathfrak{z})$ for the set of 2-coboundaries and $Z^2(\mathfrak{g},\mathfrak{z})$ for the set of 2-cocycles. f is a coboundary exactly if it vanishes on $Z_2(\mathfrak{g})$. This leads to the following description of the second $\mathfrak{z}$-valued cohomology group

$$H^2(\mathfrak{g},\mathfrak{z}) := Z^2(\mathfrak{g},\mathfrak{z})/B^2(\mathfrak{g},\mathfrak{z}) \cong \mathrm{Hom}(H_2(\mathfrak{g}),\mathfrak{z}) \to \mathrm{Hom}(Z_2(\mathfrak{g}),\mathfrak{z}).$$

The computation consists thus of a carful description of the spaces of cycles and boundaries of a current algebra $A \otimes \mathfrak{k}$. Indeed, the first step is to show that the direct sum decomposition

$$\Lambda^2(\mathfrak{g}) \cong (\Lambda^2(A) \otimes S^2(\mathfrak{k})) \oplus (A \otimes \Lambda^2(\mathfrak{k})) \oplus (I_A \otimes \Lambda^2(\mathfrak{k})),$$

where $I_A \subset S^2(A)$ is the kernel of the multiplication map, induces a corresponding decomposition of the space of 2-cycles

$$Z_2(\mathfrak{g}) \cong (\Lambda^2(A) \otimes S^2(\mathfrak{k})) \oplus (A \otimes Z_2(\mathfrak{k})) \oplus (I_A \otimes \Lambda^2(\mathfrak{k})).$$

Since two cocycles f and g define the same cohomology class if and only if they coincide on the subspace $Z_2(\mathfrak{g})$ of $\Lambda^2(\mathfrak{g})$, a cohomology class $[f] \in H^2(\mathfrak{g}, \mathfrak{z})$ is represented by three linear maps

$$f_1 : \Lambda^2(A) \otimes S^2(\mathfrak{k}) \to \mathfrak{z}, \quad f_2 : A \otimes Z_2(\mathfrak{k}) \to \mathfrak{z}, \quad \text{and} \quad f_3 : I_A \otimes \Lambda^2(\mathfrak{k}) \to \mathfrak{z},$$

satisfying $f = f_1 \oplus f_2 \oplus f_3$ on $Z_2(\mathfrak{g})$. Conversely, three such linear maps f_1, f_2 and f_3 define a cocycle if and only if $f := f_1 \oplus f_2 \oplus f_3$ vanishes on $B_2(\mathfrak{g})$. The main result of [25] is the following theorem:

Theorem 9 (Description of cocycles) *The function $f = f_1 + f_2 + f_3$ as above is a 2-cocycle if and only if the following conditions are satisfied:*

(a) $\mathrm{im}(\tilde{f}_1) \subset \mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}}$.

(b) $\tilde{f}_1(T_0(A))$ *vanishes on* $\mathfrak{k} \times \mathfrak{k}'$.

(c) $d_{\mathfrak{k}}(\tilde{f}_2(a)) = \Gamma(\tilde{f}_1(a, 1))$ *for each* $a \in A$.

(d) $\tilde{f}_3(I_A)$ *vanishes on* $\mathfrak{k} \times \mathfrak{k}'$.

Here the tilde maps $\tilde{f}_i$ for $i = 1, 2, 3$ are the original maps f_i where the $\mathfrak{k}$ arguments are moved from the domain to the range. $\mathrm{Sym}^2(\mathfrak{k}, \mathfrak{z})^{\mathfrak{k}}$ is the space of symmetric $\mathfrak{k}$-invariant bilinear maps from $\mathfrak{k}$ to $\mathfrak{z}$, $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ is the derived Lie algebra, $d_{\mathfrak{k}}$ is the coboundary operator for the Lie algebra $\mathfrak{k}$, the map Γ is the map associating to a symmetric invariant bilinear form κ the 3-cocycle $\Gamma(\kappa)(x, y, z) := \kappa([x, y], z)$, and $T_0(A)$ is the following space. We define two trilinear maps

$$T : A^3 \to \Lambda^2(A), \quad (a, b, c) \mapsto \sum_{\mathrm{cyc}} ab \wedge c := ab \wedge c + bc \wedge a + ca \wedge b$$

and

$$T_0 : A^3 \to \Lambda^2(A), \quad T_0(a, b, c) := T(a, b, c) - abc \wedge 1.$$

Then put $T_0(A) := \mathrm{span}(\mathrm{im}(T_0))$.

Cocycles of the form $f_1 \oplus f_2$, where f_1 and f_2 are not cocycles, are called *coupled*. It is a new notion which comes out naturally from our work. All coboundaries are of the form $f = f_2$, so that the cohomology class of a coupled cocycle contains only coupled cocycles.

We show further that $\mathfrak{g}$ posesses non-zero coupled 2-cocycles if and only if the image of the universal derivation $d_A : A \to \Omega^1(A)$ is non-trivial and $\mathfrak{k}$ possesses a symmetric invariant bilinear form κ for which the 3-cocycle $\Gamma(\kappa)$ is a non-zero coboundary. Calling an invariant symmetric bilinear form $\kappa \in \mathrm{Sym}^2(\mathfrak{k})^{\mathfrak{k}}$ *exact* if $\Gamma(\kappa)$ is a coboundary, this means that $\mathfrak{k}$ possesses exact invariant bilinear forms κ with $\Gamma(\kappa)$ non-zero.

We have a stock of examples of Lie algebras $\mathfrak{k}$ which possess exact invariant bilinear forms κ with $\Gamma(\kappa)$ non-zero, but we did not arrive at a satisfying classification.

3.3 Holomorphic current groups

In a third work [26] together with Karl-Hermann Neeb, we addressed the question whether the topological groups of type $\mathcal{C}^\infty(M,K)$ for a non-compact manifold M and a Lie group K and $\mathcal{O}(X,K)$ for a complex Stein manifold X and a complex Lie group K are infinite dimensional Fréchet Lie groups. The Lie algebra and model space of $\mathcal{O}(X,K)$ should be the Fréchet space $\mathcal{O}(X,\mathfrak{k})$, $\mathfrak{k}$ being the Lie algebra of K. See appendix C for precisions about infinite dimensional manifolds and Lie groups.

The link between Lie group and Lie algebra in infinite dimensions is not as tight as in the finite dimensional setting. For example, in case X possesses non-constant holomorphic functions (which is the case for a Stein manifold) and $\exp_K(\mathfrak{k}) \neq K$, we can show that the exponential map

$$\exp : \mathcal{O}(X,\mathfrak{k}) \to \mathcal{O}(X,K)$$

given by $\exp(f) := f \circ \exp_K$ for $\exp_K$ the exponential map on K, is not locally surjective. This eliminates the exponential function as a candidate for charts for the manifold structure.

We show in [26] the following theorem:

Theorem 10 *Let Σ be a connected open Riemann surface with finitely generated $\pi_1(\Sigma)$, and let K be a connected complex Banach Lie group. Then $\mathcal{O}(X,K)$ possesses a unique Fréchet Lie group structure such that the Lie algebra is $\mathcal{O}(X,\mathfrak{k})$ and the evaluation map*

$$\mathrm{ev} : \mathcal{O}(X,K) \times \Sigma \to K$$

holomorphic.

Sketch of proof. Let us sketch the proof of the theorem for the case $\Sigma = \mathbb{C}^\times$. The *logarithmic derivative* δ, cf appendix C below, can be used to establish a bijection

$$\mathcal{O}_*(\mathbb{C},K) \to \Omega^1(\mathbb{C},\mathfrak{k});$$

its inverse is the *evolution map* which associates to a Lie algebra valued differential 1-form α the solution of the ordinary differential equation

$$\begin{cases} f(0) = e \\ \delta f = \alpha f \end{cases} \qquad (3)$$

δ is injective, because we restricted to based maps $\mathcal{O}_*(\mathbb{C},K)$ which are supposed to send $0 \in \mathbb{C}$ to the unit element $e \in K$. Surjectivity of δ depends on two things: on the one hand, the image of δ is contained in the Maurer-Cartan subspace of $\Omega^1(\mathbb{C},\mathfrak{k})$, on the other hand, the problem (3) is only solvable in case α does not have a non-trivial holonomy, i.e. α is in the kernel of the *period map*

$$P : \Omega^1(X,\mathfrak{k}) \to \mathrm{Hom}(\pi_1(X),K).$$

As $\mathbb{C}$ is of dimension 1, the Maurer-Cartan equation is trivially verified for all $\mathfrak{k}$-valued 1-forms, and as $\mathbb{C}$ is 1-connected, there is no obstruction coming from periods either. The bijection permits on the one hand to put the structure of a Fréchet manifold (here actually a Fréchet space) on $\mathcal{O}_*(\mathbb{C},K)$ (and à fortiori on $\mathcal{O}(\mathbb{C},K)$), and on the other hand to transport the group structure on $\Omega^1(\mathbb{C},\mathfrak{k})$: we put

$$\alpha * \beta \,:=\, \delta(f\cdot g) \,=\, \delta(g) + \mathrm{Ad}(g)^{-1}\delta(f)$$

for $f,g\in\mathcal{O}_*(\mathbb{C},K)$ such that $\delta(f)=\alpha$ and $\delta(g)=\beta$. Observe that it is only necessary that β has a preimage under δ in order to define the product.

Now consider the same setting for $\mathcal{O}_*(\mathbb{C}^\times,K)$:

$$\mathcal{O}_*(\mathbb{C}^\times,K) \overset{\delta}{\hookrightarrow} \Omega^1(\mathbb{C}^\times,\mathfrak{k}) \overset{P}{\to} \mathrm{Hom}(\pi_1(X),K).$$

One has $P^{-1}(1)=\mathcal{O}_*(\mathbb{C}^\times,K)$. The idea is now to show that P is a *submersion*, i.e. TP has local sections, and to apply the parametrized inverse function theorem of Glöckner [11].

Local sections of TP can be obtained as follows: let $\beta\in P^{-1}(1)$, and set $\alpha_X(z)=\frac{1}{2\pi i}\frac{dz}{z}X$ for $X\in\mathfrak{k}$. The section σ_β around β is then defined by

$$\sigma_\beta:\mathfrak{k}\to\Omega^1(\mathbb{C}^\times,\mathfrak{k}),\quad X\mapsto \alpha_X * \beta.$$

This gives a section

$$U \overset{\log_U}{\to} \mathfrak{k} \overset{\sigma_\beta}{\to} \Omega^1(\mathbb{C}^\times,\mathfrak{k})$$

to $P:\Omega^1(\mathbb{C}^\times,\mathfrak{k})\to U$, because

$$P(\alpha_X*\beta) \,=\, P(\alpha_X)\cdot P(\beta) \,=\, P(\alpha_X) \,=\, \exp_K(X).$$

This shows the theorem in the special case $\Sigma=\mathbb{C}^\times$. $\square$

The obvious open question is how to take care of the Maurer-Cartan equation in order to make holomorphic current groups with higher dimensional domain into a Lie group.

4 Deformations of infinite dimensional Lie algebras

4.1 Deformations of Lie algebras of vector fields arising from families of schemes

I became interested in deformations of infinite dimensional Lie algebras by an article of Alice Fialowski and Martin Schlichenmaier [7] about non-trivial deformations of the Virasoro algebra and $\mathrm{Vect}(S^1)$ coming from taking Krichever-Novikov algebras on families of schemes. As an example, look at elliptic curves in $\mathbb{C}P^2$. They are parametrized by e_1, e_2 and e_3 and have the equation

$$Y^2Z \,=\, 4(X-e_1Z)(X-e_2Z)(X-e_3Z)$$

with $e_1 + e_2 + e_3 = 0$ and $\triangle = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_3 - e_2)^2 \neq 0$, the last condition implying non-singularity. These equations form a *family of elliptic curves* over the base $B = \{(e_1, e_2, e_3) \,|\, e_1 + e_2 + e_3 = 0, e_i \neq e_j \ \forall i \neq j\}$. Completing the base or parameter space to $\hat{B} = \{(e_1, e_2, e_3) \,|\, e_1 + e_2 + e_3 = 0\}$ admits singular cubics. Partial degenerations (e.g. $e_1 = e_2 \neq e_3$) lead to the *nodal cubic*

$$Y^2Z = 4(X - eZ)^2(X + 2eZ),$$

while overall degeneration (i.e. $e_1 = e_2 = e_3 = 0$) gives the *cuspidal cubic*

$$Y^2Z = 4X^3.$$

The nodal cubic is *stable* in the sense of Mumford-Deligne, and is therefore a point of the boundary in the Deligne-Mumford compactification of the moduli space, while the cuspidal cubic is not. Fialowski-Schlichenmaier then introduce marked points on these families of cubics, and gives explicit formulae in terms of generators and relations for the Lie algebras of vector fields on the pointed families. They show that these families give global deformations over $\mathbb{C}[t]$ of the Witt algebra $\mathrm{Vect}_{\mathrm{pol}}(S^1)$ (attained for $t = 0$) such that all Lie algebras for $t \neq 0$ are isomorphic, but non-isomorphic to the Witt algebra. One therefore gets a non-trivial deformation of an infinitesimally and formally rigid (i.e. $H^2(\mathrm{Vect}_{\mathrm{pol}}(S^1); \mathrm{Vect}_{\mathrm{pol}}(S^1)) = 0$) Lie algebra. This shows the limits of cohomology-based deformation theory for infinite dimensional Lie algebras.

My wish was to understand their construction method in a more abstract way, i.e. as a morphism between the moduli space of curves $\mathcal{M}_{g,n}$ and the moduli space of deformations of Lie algebra $\mathcal{D}ef$. It is well studied and established what the moduli space of curves $\mathcal{M}_{g,n}$ is: it is an algebraic *stack* with many interesting properties. So the first step was to construct a stack (which is unfortunately far from being algebraic) of deformations of Lie algebras $\mathcal{D}ef$. The stack $\mathcal{D}ef$ is defined on the category of affine schemes $\mathtt{Aff}/\mathbb{C}$ by a lax functor $\mathcal{D}ef$, associating to an affine scheme $U = \mathrm{Spec}(B)$ the groupoid of B-Lie algebras; here B is a unital commutative associative algebra and morphisms in the groupoid are only the isomorphisms. It follows from Grothendieck's theorem on descente fidèlement plate [13] that the pseudofunctor $\mathcal{D}ef$ satisfies the decent conditions to make it a stack (in the fpqc, fppf or étale topology).

Now there is a morphism of stacks $I : \mathcal{M}_{g,n} \to \mathcal{D}ef$ constructed exactly in the same way as Fialowski-Schlichenmaier construct their examples: given a family of projective curves with marked points over $U = \mathrm{Spec}(B)$, take out the points and take the Lie algebra of derivations on the corresponding scheme – this gives a B-Lie algebra. In order to make I a morphism, we have to restrict now to the fppf or étale topology. Observe that $\mathrm{Der}(-,-)$ on the category of algebras or $\mathrm{Vect}(-)$ on the category of manifolds is not a functor in general; you can not in general push forward or pull back a vector field by a morphism. Nevertheless, they become functors when admitting only isomorphisms in your category !

We observed that the morphism I is almost a monomorphism due to results of Siebert [28]. Indeed, Siebert's theory implies that an affine algebraic variety over an integral normal B is completely determined by its Lie algebra of derivations. For more general base schemes, this is not the case. On the other hand, on the families of smooth curves, I is a monomorphism.

We hope to use the morphism I to pullback interesting sheaves from $\mathcal{D}ef$ to $\mathcal{M}_{g,n}$, like sheaves corresponding to the Lie algebra cohomology, and to show that they live in the ring of geometric classes on $\mathcal{M}_{g,n}$ (which is reasonable from the cohomology computations). This is work in progress [37].

4.2 Deformations of $\mathfrak{m}_0$

Together with Alice Fialowski, we considered the deformation theory of the Lie algebra $\mathfrak{m}_0$ given by generators e_i, $i \geq 1$, and relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$.

In [8], we first computed the algebraic cohomology spaces $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ and $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$. As expected, they are both infinite dimensional. The story could end here, because infinite dimensional vector spaces are kind of meaningless, but there is some extra structure in our case. The Lie algebra $\mathfrak{m}_0$ is $\mathbb{N}$-graded, and therefore the cohomology spaces are. $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ carries even a bracket that makes it a graded Lie algebra. In [8], we computed this Lie algebra structure:

Theorem 11 *The bracket structure on $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ is described as follows: the commuting weight zero generators ω_1 and ω_2 act on the trivial Lie algebra generated by γ in weight 1 and the α_l for weight $l \geq 2$ as grading elements, γ has degree -1 w.r.t. ω_1, degree 1 w.r.t. ω_2, while α_l has degree l w.r.t. ω_1 and degree 0 w.r.t. ω_2.*

As for $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$, it is even worse, because the space is even infinite dimensional in every graded component. But the elements of $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$ give the infinitesimal deformations of the Lie algebra $\mathfrak{m}_0$, and thus it is a natural question to ask which infinitesimal deformations "integrate" to a true (polynomial or formal) deformation. We showed that this conditions chooses in negative or zero weight a finite number of classes, and linked our results to the known classification of of infinite dimensional $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$ and two generators over a field of characteristic zero. The main result of [8] is the following:

Theorem 12 *The true deformations of $\mathfrak{m}_0$ are finitely generated in each weight $l \leq 1$. More precisely, the space of unobstructed cohomology classes is in degree*

- *$l \leq -3$ or $l = 0$ of dimension two,*
- *$l = -2$ of dimension three,*

while there is no true deformation in weight $l = -1$. In weight $l = 0$, these are deformations to $\mathfrak{m}_1$ and L_1. In weight $l = 1$, there are exactly two such classes, while in weight $l \geq 2$, there are at least two.

We do not have more precise information about how many true deformations there are in higher positive weight. As a deformation in these weights is a true deformation if and only if all of its Massey squares are zero (as cochains !), true deformations are determined by a countable infinite system of homogeneous quadratic equations in countably infinitely many variables. We didn't succeed in determining the space of solutions of this system.

In a futur work, we will study the same questions for $\mathfrak{m}_2$ given by generators e_i, $i \geq 1$, and relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$. This should be easier, as our first computations show that there is less cohomology.

A Glossary on cocycles for Lie algebras of vector fields

Let us recall in this appendix well known cohomology computations for some Lie algebras of vector fields, and some well known cocycles, together with a construction method for cocycles. Basic references are [10] and [5].

Let us start with the computation of $H^*(W_1;\mathbb{C})$. Denote the elements of W_1 by $e_i = x^{i+1}\frac{d}{dx}$, $i = -1, 0, 1, 2, \ldots$. A general element of W_1 is a formal linear combination of the e_i. The spaces $C^p(W_1;\mathbb{C})$ are constructed from the continuous dual W_1' of W_1, with $C^1(W_1;\mathbb{C}) = W_1'$, and W_1' is naturally identified with the space of polynomials in one variable, cf [30] thm. 22.1, p. 228. Therefore, we denote its elements by ϵ_i, $i = -1, 0, 1, 2, \ldots$, dual to e_i, and a general element of W_1' is a finite linear combination of the ϵ_i. Now, the reduction to the space of *Euler-invariant cochains*, i.e. to cochains invariant with respect to the action of the Euler field e_0, cf thm. 1.5.2 p. 45 [10], permits the reduction to a finite subcomplex. This is due to the fact that e_0 acts as a grading operator on W_1 and on its dual W_1'. The subcomplex is then given by $C^1(W_1;\mathbb{C})^{e_0} = \mathbb{C}\epsilon_0$, $C^2(W_1;\mathbb{C})^{e_0} = \mathbb{C}\epsilon_1 \wedge \epsilon_{-1}$, $C^3(W_1;\mathbb{C})^{e_0} = \mathbb{C}\epsilon_1 \wedge \epsilon_0 \wedge \epsilon_{-1}$, and $C^p(W_1;\mathbb{C})^{e_0} = 0$ for $p > 3$. The coboundary operator d has as its only non-zero term on the subcomplex $d\epsilon_0 = -\frac{1}{2}\epsilon_1 \wedge \epsilon_{-1}$. Therefore, one gets $\dim H^*(W_1;\mathbb{C}) = 1$ for $* = 0, 3$ and $\dim H^*(W_1;\mathbb{C}) = 0$ otherwise.

When writing the cocycle $\epsilon_1 \wedge \epsilon_0 \wedge \epsilon_{-1}$ as a function on three formal vector fields f, g, h (identified with their coefficient functions), one gets (up to constants) the *Godbillon-Vey cocycle*

$$\theta_0(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}(0).$$

θ_0 is the evaluation at $0 \in \mathbb{R}$ of a family of cochains

$$\theta_t(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}(t) = \begin{vmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{vmatrix}.$$

θ_t and θ_s for $t \neq s$ are linked by the action of any diffeomorphism of $\mathbb{R}$ which sends s to t. The action of a Lie algebra $\mathfrak{g}$ on its cohomology is trivial, because it is given by the Lie derivative $L_X = d \circ i_X + i_X \circ d$ of an element $X \in \mathfrak{g}$ and thus L_X sends a cocycle to a coboundary. Therefore, the action of any Lie group element which can be written as a 1-parameter group is trivial. One deduces that θ_t and θ_s are cohomologuous cocycles and that the point of evaluation does not play any rôle.

Let us now comment on the cohomology of $\mathrm{Vect}(S^1)$. Using Theorem 1 together with the computation of the cohomology of the other quotient complexes Gelfand-Fuks (cf [10]) shows that

$$H^*(\mathrm{Vect}(S^1); \mathbb{C}) \,=\, \Lambda[\theta_0] \otimes S[\omega],$$

i.e. it is the tensor product of an exterior algebra in θ_0 (or any θ_t for $t \in S^1 = \mathbb{R}/\mathbb{Z}$) which is of degree 3, and a symmetric algebra in a generator ω of degree 2. ω is the *Gelfand-Fuks cocycle*

$$\omega(f,g) \,=\, \int_{S^1} \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} (t)dt.$$

It is the *integral over the fiber* of the the Godbillon-Vey cocycle θ_t in a sense which I will describe in a moment.

Let me remark that the cohomology $H^*(\mathrm{Vect}(S^1); \mathbb{C})$ is equally well described by saying that it is the singular cohomology of the topological space $\mathrm{Map}(S^1, S^3)$ in the compact-open topology. Indeed, the complex manifold X_1 is homotopically equivalent to S^3, and the corresponding bundle E on S^1 is trivial.

In order to define the procedure of *integration over the fiber*, which I learnt from Boris Shoikhet, let M be a manifold of dimension n with a system of coordinates ϕ_x in $x \in M$ which depends smoothly on x (thus M has trivial tangent bundle). Each system of coordinates ϕ_x induces a Taylor expansion morphism

$$\Phi_x : \mathrm{Vect}(M) \,\to\, W_n,$$

and therefore a morphism of complexes

$$\Phi_x^* : C^*(W_n, \mathbb{C}) \,\to\, C^*(\mathrm{Vect}(M), \mathbb{C}).$$

Given a cocycle θ of degree q on W_n, $x \mapsto \Phi_x^*\theta$ is a smooth family of cohomologuous cocycles on $\mathrm{Vect}(M)$ (once again because of the triviality of the action of diffeomorphisms). Taking the de Rham differential of this function $\Phi_x^*\theta$ with values in cocycles, $d_{\mathrm{dR}}\Phi_x^*\theta$ is a coboundary, say $d_{\mathrm{dR}}\Phi_x^*\theta = d^{\mathbb{C}}\omega$ for some 1-form with values in $(q-1)$-cochains ω ($d^{\mathbb{C}}$ being the Lie algebra coboundary operator with values in the trivial module $\mathbb{C}$). Taking a 1-cycle σ on M, the integral $\int_\sigma \omega$ is by definition the fiber integral of θ. It is indeed a cocycle by the de Rham theorem:

$$d^{\mathbb{C}} \int_\sigma \omega \,=\, \int_\sigma d^{\mathbb{C}}\omega \,=\, \int_\sigma d_{\mathrm{dR}} \Phi_x^*\theta \,=\, \int_{\partial\sigma} \Phi_x^*\theta \,=\, 0.$$

Therefore, the equation (2) $d^{\mathbb{C}}\alpha = d_{\mathrm{dR}}\theta_t$ for the 2-cocycle with values in 1-forms (or 1-densities, cf section 1.3)

$$\alpha(f,g) \;=\; \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}$$

means that the Gelfand-Fuks cocycle ω is the fiber integral of the Godbillon-Vey cocycle.

Observe that the cocycles θ_0 and ω in $H^*(\mathrm{Vect}(S^1);\mathbb{C})$ are not of the same nature: θ_0 is a *local* cocycle, i.e. the support (cf section 1.1) of $\theta(f,g,h)$ is included in the intersection of the supports of f, g and h. By a well known theorem of Peetre, local operators are differential operators. The product in $C^*(\mathrm{Vect}(M),\mathbb{C})$ of local cocycles is local, and the local cochains form even a subcomplex.

On the other hand, the support of ω is the whole S^1, because it is an integral. It is the integral of a local cocycle, therefore we call it diagonal. In general a cocycle $c \in C^q(\mathrm{Vect}(M);\mathbb{C})$ is called *diagonal* in case $c(X_1,\ldots,X_q) = 0$ whenever $\bigcap_{i=1}^q \mathrm{supp}(X_i) = \emptyset$, i.e. in case the generalized section c of the vector bundle $\bigotimes^q TM$ on M^q is concentrated on the diagonal $\triangle \subset M^q$ (cf section 1.1).

In [19], Lodder computes the continuous Leibniz cohomology (cf section 1.4) of the Lie algebra W_1. It turns out to be

$$HL^*(W_1;\mathbb{C}) \;=\; \Lambda[\theta_0]\otimes T[\beta],$$

the tensor product of an exterior algebra on θ_0 with a tensor algebra on a generator β of degree 4, for which an explicit formula looks like:

$$\beta(l,f,g,h) \;=\; l'(0)\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}(0).$$

One sees that $\beta = \epsilon_0 \otimes \theta_0$, for ϵ_0 dual to the Euler vector fields e_0. The first new generator γ_1 is then given by the fiber integral of β:

$$\gamma_1(l,f,g) \;=\; \int_{S^1} l'(t)\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}(t)\; dt.$$

B Glossary on some subjects from homological algebra

For a Lie algebra $\mathfrak{g}$, the homology of $\mathfrak{g}$ and the cohomology of $\mathfrak{g}$ are defined as the derived functors of the functors of coinvariants and invariants on the category of $U\mathfrak{g}$-modules. For explicit computations, Chevalley and Eilenberg invented the Chevalley-Eilenberg complex as an explicit resolution of the ground field $\mathbb{C}$. It is given by a Koszul-complex type differential on the collection of vector spaces $U\mathfrak{g}\otimes\Lambda\mathfrak{g}$. Applying the functor $\mathrm{Hom}_{U\mathfrak{g}}(-,\mathfrak{z})$ for some fixed $U\mathfrak{g}$-module $\mathfrak{z}$ to this resolution gives a complex computing the cohomology with values in $\mathfrak{z}$. As the morphisms are $U\mathfrak{g}$-equivariant, one may simplify to get the *Chevalley-Eilenberg complex* $(\mathrm{Hom}_{\mathbb{C}}(\Lambda\mathfrak{g},\mathfrak{z}),d)$. For the trivial module $\mathfrak{z}=\mathbb{C}$, the differential reduces to the formula given in section 1, the only difference being that for the Gelfand-Fuks cohomology the cochains are *continuous* linear maps for a topological Lie algebra $\mathfrak{g}$.

There are some standard interpretations of the low dimensional cohomology spaces. Keep in mind that these interpretations do not change in the topological framework of cohomology; the only difference is that one takes *topologically split* exact sequences, i.e. sequences that are split as sequences of topological vector spaces (i.e. images and kernels have closed supplementary subspaces or equivalently, all maps have continuous sections) $H^2(\mathfrak{g};\mathfrak{z})$ classifies the *abelian extensions* of $\mathfrak{g}$ by $\mathfrak{z}$, i.e. the short exact sequences of Lie algebras

$$0\to\mathfrak{z}\xrightarrow{i}\mathfrak{e}\xrightarrow{\pi}\mathfrak{g}\to 0$$

such that $\mathfrak{z}$ is (identified to) an abelian subalgebra of $\mathfrak{e}$. The bracket on $\mathfrak{e}$ is written in its most general form as

$$[(a,x),(b,y)]\;=\;(x\cdot b-y\cdot a+\alpha(x,y),[x,y])$$

by fixing an identification $\mathfrak{e}\cong\mathfrak{z}\oplus\mathfrak{g}$ as (topological) vector spaces and with $a,b\in\mathfrak{z}$ and $x,y\in\mathfrak{g}$.

The only data in a given abelian extension which is not specified by the Lie algebra $\mathfrak{g}$ and the $\mathfrak{g}$-module $\mathfrak{z}$, is the (continuous) 2-cocycle α which characterizes the extension up to adding a coboundary. Indeed, $H^2(\mathfrak{g};\mathfrak{z})$ is in bijection to the equivalence classes of abelian extensions, the map being given by associating to an extension its cocycle α, and to a cocycle α the bracket on $\mathfrak{z}\oplus\mathfrak{g}$ given above. Here, an extension

$$0\to\mathfrak{z}\to\mathfrak{e}\to\mathfrak{g}\to 0$$

is called *equivalent* to an extension

$$0\to\mathfrak{z}\to\mathfrak{e}'\to\mathfrak{g}\to 0$$

if there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "id_V"] & \mathfrak{e} \arrow[r, "\pi"] \arrow[d, "\psi"] & \mathfrak{g} \arrow[r] \arrow[d, "id_{\mathfrak{g}}"] & 0 \\ 0 \arrow[r] & V \arrow[r, "i'"] & \mathfrak{e}' \arrow[r, "\pi'"] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

The abelian extension is called a *central extension* in case $\mathfrak{z}$ is a trivial $\mathfrak{g}$-module. For example, the *Virasoro algebra* is the central extension of $\mathrm{Vect}(S^1)$ by the Gelfand-Fuks cocycle ω. As $H^2(\mathrm{Vect}(S^1);\mathbb{C})$ is 1-dimensional, the Virasoro algebra has the special property of being the *universal central extension* of $\mathrm{Vect}(S^1)$, i.e. every central extension of $\mathrm{Vect}(S^1)$ factorizes by the Virasoro algebra.

The problem of universality for central extensions of the current algebra $A \otimes \mathfrak{k}$ for a simple finite dimensional complex Lie algebra $\mathfrak{k}$ and a commutative unital Fréchet algebra A has been solved by Maier in [20]. He shows explicitly that given a continuous 2-cocycle ω' with values in a Fréchet space $\mathfrak{z}$ factorises as $\omega' = \xi \circ \omega_A$ for some continuous map $\xi : \Omega^1(A) / \overline{dA} \to \mathfrak{z}$ and the universal cocycle ω_A given by

$$\omega_A(a \otimes x, b \otimes y) \,=\, \kappa(x, y)\, a\, d_A b.$$

Here κ (also earlier denoted $\langle , \rangle$) is the Killing form on $\mathfrak{g}$, and $(\Omega^1(A), d_A)$ is the *Fréchet A-module of Kähler differentials.*

The universal property of $(\Omega^1(A), d_A)$ is that for every Fréchet A-module M and every continuous derivation $D : A \to M$, there is a unique continuous A-linear map $\phi : \Omega^1(A) \to M$ such that $D = \phi \circ d_A$. It can be constructed as the module of Kähler differentials, i.e. if I_A is the kernel of the multiplication map $\mu : A \otimes A \to A$ (where tensor products carry the π-topology), then $\Omega^1(A) = I_A / \overline{I_A^2}$. One has to complete in order to make the quotient a Fréchet space. The universal map $d_A : A \to \Omega^1(A)$ is defined by $d_A(a) := [a \otimes 1 - 1 \otimes a]$, the class being taken in the quotient.

Let us now pass to 3-cohomology. In the same way as 2-cohomology classifies abelian extensions, 3-cohomology classifies *crossed modules.* A crossed module is a morphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action η of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations such that

(a) $\mu(\eta(n) \cdot m) \,=\, [n, \mu(m)]$ for all $n \in \mathfrak{n}$ and all $m \in \mathfrak{m}$,

(b) $\eta(\mu(m)) \cdot m' \,=\, [m, m']$ for all $m, m' \in \mathfrak{m}$.

To each crossed module of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$, one associates a four term exact sequence

$$0 \to V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \to 0$$

where $ker(\mu) =: V$ and $\mathfrak{g} := coker(\mu)$.

In the framework of (infinite dimensional) locally convex Lie algebras and locally convex modules, we suppose once again that the above sequence is topologically split (i.e. all images and kernels are closed and topologically direct summands).

- By (a), $\mathfrak{g}$ is a Lie algebra, because $im\,(\mu)$ is an ideal.
- By (b), V is a central Lie subalgebra of $\mathfrak{m}$, and in particular abelian.
- By (a), the action of $\mathfrak{n}$ on $\mathfrak{m}$ induces a structure of a $\mathfrak{g}$-module on V.
- Note that in general $\mathfrak{m}$ and $\mathfrak{n}$ are not $\mathfrak{g}$-modules.

Two crossed modules $\mu : \mathfrak{m} \to \mathfrak{n}$ (with action η) and $\mu' : \mathfrak{m}' \to \mathfrak{n}'$ (with action η') such that $ker(\mu) \,=\, ker(\mu') =: V$ and $coker(\mu) \,=\, coker(\mu') =: \mathfrak{g}$ are called *elementary equivalent* if there are morphisms of Lie algebras $\phi : \mathfrak{m} \to \mathfrak{m}'$ and $\psi : \mathfrak{n} \to \mathfrak{n}'$ such that they are compatible with the actions, meaning:

$$\phi(\eta(n) \cdot m) \ = \ \eta'(\psi(n)) \cdot \phi(m) \quad \forall n \in \mathfrak{n} \ \forall m \in \mathfrak{m},$$

and such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "id_V"] & \mathfrak{m} \arrow[r, "\mu"] \arrow[d, "\phi"] & \mathfrak{n} \arrow[r, "\pi"] \arrow[d, "\psi"] & \mathfrak{g} \arrow[r] \arrow[d, "id_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r, "i'"] & \mathfrak{m}' \arrow[r, "\mu'"] & \mathfrak{n}' \arrow[r, "\pi'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Equivalence of crossed modules is the equivalence relation generated by the elementary equivalence.

Now let $\mathfrak{g}$ be a topological Lie algebra. One result from [36] is:

Theorem 13 *Denote by* $\mathrm{crmod}_{\mathrm{top}}(\mathfrak{g}, V)$ *the abelian group of topologically split crossed modules and by* $H^3(\mathfrak{g}, V)$ *the abelian group of continuous cohomology classes. Suppose that there is a topologically split exact sequence of* $\mathfrak{g}$*-modules*

$$0 \to V \to W \to U \to 0$$

such that $H^3(\mathfrak{g}, W) = 0$.
Then there is an isomorphism of abelian groups

$$b : \mathrm{crmod}_{\mathrm{top}}(\mathfrak{g}, V) \ \cong \ H^3(\mathfrak{g}, V).$$

The abelian group structure on the set of crossed modules is given by the *Baer sum.* In the non-topological setting, this theorem is usually attribued to Gerstenhaber. In this case, one does not have to suppose the existence of any $\mathfrak{g}$-module as in the statement above as it is guaranteed by the fact that there are enough injectives.

C Glossary on infinite dimensional manifolds

Let us recommend the survey article [23] as source on infinite dimensional manifolds and Lie groups.

A *Lie group* is simply a topological group which is a manifold such that the multiplication and the inversion maps are smooth. Therefore, in order to define infinite dimensional Lie groups, it suffices to define infinite dimensional manifolds. We will do that in the framework of Fréchet spaces, although locally convex Hausdorff topological vector spaces would do. A *Fréchet space* is a complete metrizable locally convex Hausdorff topological vector space.

In order to do calculus in a Fréchet space, we must have the notion of derivative. It is curious to note that the notion of *Fréchet derivative* (which is the notion of total differentiability) is not the one most suitable for our purpose:

it presupposes a good notion of continuity on the spaces of operators. This is given by the operator norm, in the setting of Banach spaces, but there is no satisfactory notion beyond Banach spaces. Therefore we stay with the *Gâteaux derivative* (which is the notion of directional derivative).

Let U be an open set in a Fréchet space E, and $f : U \to F$ be a Fréchet space valued continuous function on U. Consider

$$df(x)(h) \;=\; \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}.$$

In case this limit exists for each $h \in E$ and each $x \in U$ and defines a continuous function $df : U \times E \to F$, f is said to be $\mathcal{C}^1$. By iteration, the concept of a $\mathcal{C}^\infty$ or smooth function is defined. The crucial point for defining manifolds is the chain rule; it holds for $\mathcal{C}^1$ maps between locally convex spaces, and this is more than enough for us here. A $\mathcal{C}^\infty$ map $f : E \to F$ between complex Fréchet spaces is called *holomorphic* in case $df(x) : E \to F$ is complex linear. Between Fréchet spaces, f is holomorphic if and only if it is *complex analytic*, i.e. approachable by polynomials as a pointwise limit.

Let us also comment on the *compact open topology* for mapping spaces like $\mathcal{O}(X,K)$. It in generated by the open sets

$$W(C,O) \;=\; \{f : X \to K \,|\, f(C) \subset O\}$$

for a compact set C in X and an open set O in K. In order to show that the compact open topology is indeed a group topology on $\mathcal{O}(X,K)$, we use prop. 1, Ch. III, §1.2 of [4] which shows that given a filter $\mathcal{F}$ on a group G which satisfies

- $\bigcap \mathcal{F} = 1$
- $\forall U \in \mathcal{F}\ \exists V \in \mathcal{F}$: $VV \subset U$
- $\forall U \in \mathcal{F}\ \exists V \in \mathcal{F}$: $V^{-1} \subset U$
- $\forall U \in \mathcal{F}\ \exists g \in G$, $\exists V \in \mathcal{F}$: $gVg^{-1} \subset U$

there is a unique group topology on G such that $\mathcal{F}$ is the filter of neighborhoods of 1.

A subset $N \subset M$ of a Fréchet manifold M modelled on the Fréchet space E is called a *split submanifold* in case there exists a split subspace $F \subset E$ (i.e. having a closed supplement) and a chart (ϕ, U) of M such that $\phi(U \cap N) = \phi(U) \cap F$. In theorem 10, we show that $\mathcal{O}_*(\Sigma, K)$ is a split submanifold of the Fréchet space $\Omega^1(\Sigma, \mathfrak{k})$.

For this, we need the logarithmic derivative of a function $f \in \mathcal{O}(X,K)$. It is defined using the *Maurer-Cartan form* $\kappa \in \Omega^1(X, \mathfrak{k})$ on K which is the unique left invariant 1-form on K such that $\kappa(X) = X$ for each $X \in \mathfrak{k}$ (seen as a left invariant vector field). The *logarithmic derivative* of $f \in \mathcal{O}(X,K)$ is then defined by $\delta(f) = f^*(\kappa) \in \Omega^1(X, \mathfrak{k})$. It satisfies the product rule

$$\delta(f \cdot g) \;=\; \mathrm{Ad}(g)^{-1}\delta(f) + \delta(g),$$

and the *Maurer-Cartan equation*, which I express here for a general 1-form ω as

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

The bracket of Lie algebra valued 1-forms α, β is as usual the Lie algebra valued 2-form

$$[\alpha, \beta](v, w) \ = \ [\alpha(v), \beta(w)] - [\alpha(w), \beta(v)]$$

for tangent vectors v, w.

References

[1] H. Behnke, K. Stein, Entwicklung analytischer Funktionen auf Riemannschen Flächen. Math. Ann. **120** (1948) 430–461

[2] R. Bott, G. Segal, The cohomology of the vector fields on a manifold. Topology **16** (1977) 285–298

[3] S. Bouarroudj, H. Gargoubi, Projectively invariant cocycles of holomorphic vector fields on an open Riemann surface. Tokyo J. Math. **25**, 1 (2002) 33–40

[4] N. Bourbaki, Topologie générale. Hermann Paris 1971

[5] B. L. Feigin, D. B. Fuks, Cohomologies of Lie groups and Lie algebras. Lie groups and Lie algebras, II, 125–223, Encyclopaedia Math. Sci., 21, Springer, Berlin, 2000

[6] Y. Félix, J.-C. Thomas, Rational Betti numbers of configuration spaces. Topology and its Appl. **102** (2000) 139–149

[7] A. Fialowski, M. Schlichenmaier, Global deformations of the Witt algebra of Krichever-Novikov type. Commun. Contemp. Math. Vol. **5**, No. 6 (2003) 921–945

[8] A. Fialowski, F. Wagemann, Cohomology and deformations of the infinite dimensional filiform Lie algebra $\mathfrak{m}_0$, math.RT/0703383, accepted in J. of Algebra

[9] A. Frabetti, F. Wagemann, On the Leibniz cohomology of vector fields. Ann. Global Anal. Geom. **21**, No. 2 (2002) 177–190

[10] D. Fuks, Cohomology of infinite dimensional Lie algebras. Nauka Moscow 1984

[11] H. Glöckner, Implicit functions from topological vector spaces to Banach spaces, Israel Journal Math. **155** (2006) 205–

[12] J. Grabowski, Isomorphisms and ideals of the Lie algebra of vector fields. Invent. Math. **50** (1978) 13–33

[13] A. Grothendieck, M. Raynaud, Revêtements étales et groupe fondamental. (SGA 1) Exposés I à XIII séminaire de géométrie algébrique du Bois Marie 1960/61 (SGA 1); augmenté de deux exposés de M. Raynaud SMF Paris 2003

[14] A. Haddi, Homologie des algèbres de Lie étendues à une algèbre commutative. Comm. in Alg. **20**,4 (1992) 1145–1166

[15] A. Haefliger, Sur la cohomologie de l'algèbre de Lie des champs de vecteurs. Ann. Sci. ENS 4ème série **9** (1976) 503–532

[16] J. Huebschmann, Poisson cohomology and quantization. J. Reine Angew. Math. **408** (1990) 57–113

[17] N. Kawazumi, On the complex analytic Gelfand-Fuks cohomology of open Riemann surfaces. Ann. Inst. Fourier Grenoble **43**,3 (1993) 655–712

[18] C. Laurent-Gengoux, F. Wagemann, Obstruction classes of crossed modules of Lie algebroids and Lie groupoids linked to existence of principal bundles, math.AT/0611226, submitted

[19] J. Lodder, Leibniz cohomology for differentiable manifolds. Ann. Inst. Fourier Grenoble **48**, 1 (1998) 73–95

[20] P. Maier, Central extensions of topological current algebras. In: Geometry and Analysis on Finite- and Infinite-dimensional Groups, A. Strasburger et al. (eds.) Banach Center Publications **55**, Wrszawa (2002) 61–76

[21] D. Millionshchikov, Krichever-Novikov algebras and the cohomology of the algebra of meromorphic vector fields, AMS Translations Ser. 2 **179** (1997) 101–108

[22] K.-H. Neeb, Non-abelian extensions of topological Lie algebras. Comm. Algebra **34** (2006) no. 3, 991–1041

[23] K.-H. Neeb, Towards a Lie theory of locally convex groups. Jpn. J. Math. **1**,2 (2006) 291–468

[24] K.-H. Neeb, F. Wagemann, Universal central extension of holomorphic current algebras. Manuscripta Math. **112** (2003) 441–458

[25] K.-H. Neeb, F. Wagemann, The second cohomology of current algebras of general Lie algebras. accepted at Canadian J. of Math.

[26] K.-H. Neeb, F. Wagemann, Lie group structures on the group of smooth and holomorphic maps on non-compact manifolds. math.DG/0703460, submitted

[27] G. Rinehart, Differential forms over general commutative algebras. Trans. AMS **108** (1963) 195–222

[28] T. Siebert, Lie algebras of derivations and affine algebric geometry over fields of characteristic 0. Math. Ann. **305** (1996) 271–286

[29] S. Skryabin, Degree one cohomology for the Lie algebras of derivations. Lobachevskii J. Math. **14** (2004) 86–123

[30] F. Trèves, Topological vector spaces, distributions and kernels. Academic Press 1967

[31] F. Wagemann, Some Remarks on the Cohomology of Krichever-Novikov Algebras. Lett. Math. Phys. **47** (1999) 173–177

[32] F. Wagemann, Differential Graded Cohomology and Lie Algebras of Holomorphic Vector Fields. Comm. Math. Phys. **208** (1999) 521–540

[33] F. Wagemann, A Two Dimensional Analogue of the Virasoro Algebra. J. Geom. Phys **36** (2000) 103–116

[34] F. Wagemann, A Crossed Module representing the Godbillon-Vey Cocycle. Lett. Math. Phys. **51** (2000) 293–299

[35] F. Wagemann, Explicit Formulae for Cocycles of Holomorphic Vector fields with values in λ-Densities. J. Lie Theory, Vol. **11**, 1 (2001) 173–184

[36] F. Wagemann, On Lie algebra crossed modules. Comm. Algebra 34 (2006) no. 5, 1699-1722

[37] F. Wagemann, Deformations of Lie algebras of vector fields arising from families of schemes. http://arxiv.org/abs/0707.4054, submitted

[38] P. Zusmanovich, The second homology group of current Lie algebras. SMF Astérisque **226** (1994) 435–452