Cohomology of Lie algebras and crossed modules

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Introduction

These are the lecture notes for my lectures from the 16th of July to the 20th of july 2018 for the EAUMP-ICTP School and Workshop on Homological Methods in Algebra and Geometry II.

The goal of this series of lectures is an introduction to the cohomology of Lie algebras with an emphasis on crossed modules of Lie algebras. Every algebraic structure comes with a cohomology theory. The cohomology spaces are certain invariants which may serve to reflect the structure of the Lie algebra in question or to distinguish Lie algebras, but which may also enable to construct new Lie algebras as central or abelian extensions by the help of 2-cocycles. Central extensions play an important role in physics, because they correspond to *projective* representations of the non-extended Lie algebra which are important in quantum theory.

Crossed modules are an algebraic structure which exists not only for Lie algebras and groups, but in many more algebraic contexts. Their first goal is to represent 3-cohomology classes. Next, they are related to the question of existence of an extension for a given outer action (we will not dwell on this aspect of the theory). Another important feature is that crossed modules of Lie algebras are the *strict* Lie 2-algebras and thus give a hint towards the categorification of the notion of a Lie algebra. The study of the categorification of algebraic notions is rather new and currently of increasing interest, also in connection with physics (higher gauge theory) and representation theory.

Concerning references, a standard reference on homological algebra and the cohomology of Lie algebras is [5]. References on crossed modules of Lie algebras include [2], [3] and [4]. Crossed modules of Lie algebras seen as strict Lie 2-algebras appeared in [1].

1 Homological algebra

Here we present some preliminaries on homological algebra over a field k. Homological algebra is much simpler over a field k than over a general commutative ring. Indeed, over k, all exact sequences split as sequences of k-vector spaces, while there are non-splitting exact sequences over $\mathbb{Z}$ like for example

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Definition 1.1. A complex (of k-vector spaces) (C^*, d_*) is a sequence of k-linear maps

$$C_0 \stackrel{d_0}{\to} C_1 \stackrel{d_1}{\to} \dots C^{r-1} \stackrel{d_{r-1}}{\to} C^r \stackrel{d_r}{\to} C^{r+1} \to \dots$$

for $0 \leq r$ such that $d_r \circ d_{r-1} = 0$ for all $r \geq 1$.

Remark 1.2. The fact that $d_{r-1} : C^{r-1} \to C^r$ is expressed as saying that d_{r-1} increases degree or that (C^*, d_*) is a cohomological complex or a cochain complex. There is a variant where d_{r-1} diminishes degree and this variant is then called a homological complex or chain complex.

Definition 1.3. The cohomology of a complex (C^*, d_*) is by definition the sequence of vector spaces $H^*(C^*, d_*)$ where

$$H^p(C^*, d_*) = \frac{\ker (d_p : C^p \to C^{p+1})}{\operatorname{im} (d_{p-1} : C^{p-1} \to C^p)}.$$

Elements of $\ker (d_p : C^p \to C^{p+1})$ are called p-cocycles of C^*, elements of $\operatorname{im} (d_{p-1} : C^{p-1} \to C^p)$ are called p-coboundaries of C^*. The subspace of p-cocycles is denoted $Z^p(C^*, d_*)$, and the subspace of p-coboundaries is denoted $B^p(C^*, d_*)$.

Remark 1.4. Observe that $H^p(C^*, d_*)$ is well-defined, because $\operatorname{im} (d_{p-1} : C^{p-1} \to C^p) \subset \ker (d_p : C^p \to C^{p+1})$ by the requirement $d_p \circ d_{p-1} = 0$. Observe that if the sequence

$$C_0 \stackrel{d_0}{\to} C_1 \stackrel{d_1}{\to} \dots C^{r-1} \stackrel{d_{r-1}}{\to} C^r \stackrel{d_r}{\to} C^{r+1} \to \dots$$

is exact (i.e. $\operatorname{im} (d_{p-1} : C^{p-1} \to C^p) = \ker (d_p : C^p \to C^{p+1})$ for all $p \geq 1$), then $H^p(C^*, d_*) = 0$ for all $p \geq 0$. Therefore the cohomology measures the deviation from exactness of a complex.

Definition 1.5. A morphism of complexes $\varphi^* : (C^*, d_*^C) \to (D^*, d_*^D)$ is a sequence of k-linear maps $\varphi^p : C^p \to D^p$ such that for all $p \geq 0$, the diagrams

$$\begin{tikzcd} C^p \arrow[r, "\varphi^p"] \arrow[d, "d_p^C"] & D^p \arrow[d, "d_p^D"] \\ C^{p+1} \arrow[r, "\varphi^{p+1}"] & D^{p+1} \end{tikzcd}$$

are commutative. Here we have written d_p^C for the differential belonging to the complex (C^*, d_*^C) and d_p^D for the one belonging to (D^*, d_*^D), but we will denote generically all differentials by d_p or even d in the following in order to lighten the notation.

Lemma 1.6. *A morphism of complexes $\varphi^* : (C^*, d_*) \to (D^*, d_*)$ induces a well-defined k-linear map $H^p(\varphi) : H^p(C^*, d_*) \to H^p(D^*, d_*)$ in cohomology for all $p \geq 0$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
C^{p-1} \arrow[r, "d"] \arrow[d, "\varphi^{p-1}"] & C^p \arrow[r, "d"] \arrow[d, "\varphi^p"] & C^{p+1} \arrow[d, "\varphi^{p+1}"] \\
D^{p-1} \arrow[r, "d"] & D^p \arrow[r, "d"] & D^{p+1}
\end{tikzcd}$$

Therefore, φ^p sends cocycles to cocycles and coboundaries to coboundaries. The map $H^p(\varphi) : H^p(C^*, d_*) \to H^p(D^*, d_*)$ is defined by $H^p(\varphi)[c] := [\varphi(c)]$, where c is a p-cocycle and $[c]$ denotes its class in $H^p(C^*, d_*)$, while $[\varphi(c)]$ denotes the class of the p-cocycle $\varphi(c)$ in $H^p(D^*, d_*)$. $\square$

Definition 1.7. A homotopy between morphisms of complexes $\psi^*, \varphi^* : (C^*, d_*) \to (D^*, d_*)$ is a sequence of k-linear maps $h_p : C^p \to D^{p-1}$ for all $p \geq 1$ such that

$$\phi^p - \varphi^p = h_{p+1} \circ d_p^C + d_{p-1}^D \circ h_p$$

for all $p \geq 1$.

Lemma 1.8. *If there exists a homotopy h_* between two morphisms of complexes $\psi^*, \varphi^* : (C^*, d_*) \to (D^*, d_*)$, then the induced k-linear maps $H^p(\varphi)$ and $H^p(\psi)$ coincide for all $p \geq 0$.*

Proof. For all $c \in C^p$, we have

$$\phi^p(c) - \varphi^p(c) = h_{p+1} \circ d_p^C(c) + d_{p-1}^D \circ h_p(c).$$

Therefore we obtain for a p-cocycle $c \in Z^p(C^*, d_*)$

$$\phi^p(c) - \varphi^p(c) = d_{p-1}^D \circ h_p(c).$$

This implies that $H^p(\varphi)[c] = H^p(\psi)[c]$. $\square$

Definition 1.9. A contracting homotopy is a homotopy between the morphisms of complexes $\mathrm{id}_C : (C^*, d_*) \to (C^*, d_*)$ and the zero map $0 : (C^*, d_*) \to (C^*, d_*)$.

Lemma 1.10. *If there exists a contracting homotopy of the complex (C^*, d_*), then its cohomology vanishes, i.e. $H^p(C^*, d_*) = 0$ for all $p \geq 0$.*

Proof. Exercise. $\square$

Definition 1.11. A short exact sequence of complexes is a sequence of morphisms of complexes

$$0 \to C_1^* \xrightarrow{\varphi^*} C_2^* \xrightarrow{\psi^*} C_3^* \to 0$$

such that for all $p \geq 0$, the sequence of k-vector spaces

$$0 \to C_1^p \xrightarrow{\varphi^p} C_2^p \xrightarrow{\psi^p} C_3^p \to 0$$

is exact.

Theorem 1.12. *A short exact sequence of complexes*

$$0 \to C_1^* \xrightarrow{\varphi^*} C_2^* \xrightarrow{\psi^*} C_3^* \to 0$$

induces a long exact sequence in cohomology, i.e. there exists a k-linear map $\partial^p : H^p(C_3^*) \to H^{p+1}(C_1^*)$, called the connecting homomorphism, such that the sequence of k-vector spaces

$$\dots \xrightarrow{\partial^{p-1}} H^p(C_1^*) \xrightarrow{H^p(\varphi)} H^p(C_2^*) \xrightarrow{H^p(\psi)} H^p(C_3^*) \xrightarrow{\partial^p} H^{p+1}(C_1^*) \to \dots$$

is exact.

Proof. Exercise. □

Lemma 1.13 (Five Lemma)**.** *Suppose there is a commutative diagram of k-vector spaces*

$$\begin{tikzcd}
0 \arrow[r] & V' \arrow[r, "i"] \arrow[d, "\varphi'"] & V \arrow[r, "\pi"] \arrow[d, "\varphi"] & V'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & W' \arrow[r, "i'"] & W \arrow[r, "\pi'"] & W'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows and such that φ' and φ'' are isomorphisms. Then φ is an isomorphism.

Proof. Exercise. □

2 Lie algebras

Starting from here, we suppose that the characteristic of the field k is different from 2.

Definition 2.1. A Lie algebra is a k-vector space $\mathfrak{g}$ with a k-bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the bracket, such that

(a) $[,]$ is antisymmetric, i.e. for all $x, y \in \mathfrak{g}$: $[x, y] = -[y, x]$.

(b) $[,]$ satisfies the Jacobi identity, i.e. for all $x, y, z \in \mathfrak{g}$:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Example 2.2. (a) The most basic example of a Lie algebra is the Lie algebra A_{Lie} underlying an associative algebra A. By definition, A_{Lie} is the k-vector space A with the bracket $[a, b] := ab - ba$ for all $a, b \in A$. This applies for example to the associative algebra of $n \times n$-matrices $M_n(k)$ over the field k, or to the associative algebra of endomorphisms $\mathrm{End}(V)$ of a k-vector space V. In the first case, the corresponding Lie algebra is denoted $\mathfrak{gl}_n(k)$, in the second case, it is denoted $\mathfrak{gl}(V)$.

(b) Another different way of associating to an associative algebra A a Lie algebra is to consider its Lie algebra of derivations $\mathrm{Der}(A)$. A derivation of A is a linear map $f : A \to A$ such that $f(ab) = af(b) + f(a)b$ for all $a, b \in A$. The composition of two derivations f_1, f_2 is in general not a derivation, but $[f_1, f_2] := f_1 \circ f_2 - f_2 \circ f_1$ is easily verified to be a derivation. This bracket renders $\mathrm{Der}(A)$ a k-Lie algebra.

(c) This last example has a geometric analogue: The vector fields on a smooth manifold (or on a smooth algebraic variety) form a k-Lie algebra, where we suppose $k = \mathbb{R}$ or $\mathbb{C}$ for smooth manifolds. In case $M = G$ a Lie group (like the Lie group $Gl_n(k)$, $Sl_n(k)$, $SO_n(\mathbb{R})$, or $Sp_n(\mathbb{R})$), the left-invariant vector fields form a Lie subalgebra which leads to the Lie algebra of a Lie group (in the examples, this gives $\mathfrak{gl}_n(k)$, $\mathfrak{sl}_n(k)$, $\mathfrak{so}_n(\mathbb{R})$ or $\mathfrak{sp}_n(\mathbb{R})$). The difference is that while the Lie algebra of left-invariant vector fields on a Lie group is a finite-dimensional Lie algebra, the Lie algebra of all smooth vector fields on any smooth manifold is an infinite-dimensional Lie algebra.

Definition 2.3. A k-linear map $f : \mathfrak{g} \to \mathfrak{h}$ between k-Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is called a Lie algebra morphism in case for all $x, y \in \mathfrak{g}$, we have

$$f([x, y]) = [f(x), f(y)].$$

Example 2.4. Beyond the zero map and the identity $\mathrm{id} : \mathfrak{g} \to \mathfrak{g}$, a more interesting example is the trace

$$\mathrm{tr} : \mathfrak{gl}_n(k) \to k.$$

The 1-dimensional k-vector space k carries here the abelian or trivial Lie algebra structure, i.e. the bracket map is the zero map. The trace is a Lie algebra homomorphism, because the trace of a commutator is zero. This identifies $\mathfrak{sl}_n(k)$ as the kernel of the trace map.

Definition 2.5. A sub vector space $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra in case the bracket in $\mathfrak{g}$ of two elements in $\mathfrak{h}$ lies still in $\mathfrak{h}$: For all $x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$. It is called an ideal, in case for all $x \in \mathfrak{g}$ and all $y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

Lemma 2.6. *The kernel of a morphism of Lie algebras is an ideal.*

In case $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, the quotient vector space $\mathfrak{g}/\mathfrak{h}$ carries an induced bracket and becomes a Lie algebra. In the above example, $\mathfrak{sl}_n(k)$ is an ideal and the quotient vector space becomes the trivial Lie algebra k.

Definition 2.7. Let V be a k-vector space and $\mathfrak{g}$ a k-Lie algebra. V is called a $\mathfrak{g}$-module in case there exists a morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. Denoting the action more simply by $\phi(x)(v) =: x \cdot v$, the property to be a morphism of Lie algebras reads for all $x, y \in \mathfrak{g}$ and all $v \in V$

$$x \cdot (y \cdot v)) - y \cdot (x \cdot v)) = [x, y] \cdot v.$$

Example 2.8. It is easy to verify that A-modules for an associative algebra A become A_{Lie}-modules. For example, the $M_n(k)$-module k^n becomes a $\mathfrak{gl}_n(k)$-module (where the action is given by the action of a matrix on a vector).

3 Cohomology of Lie algebras

Let $\mathfrak{g}$ be a k-Lie algebra and V be a $\mathfrak{g}$-module. We associate to this data a complex of k-vector spaces $(C^*(\mathfrak{g}, V), d_*)$. Namely, we set

$$C^p(\mathfrak{g}, V) = \mathrm{Alt}(\mathfrak{g}^p, V)$$

for $p \geq 1$ and $C^0(\mathfrak{g}, V) = V$. Here $\mathrm{Alt}(W^n, V)$ for two vector spaces V, W denotes the space of k-multilinear, alternating maps from W^n to V. These multilinear maps $c \in \mathrm{Alt}(\mathfrak{g}^p, V$ are antisymmetric in adjacent entries, i.e. for all $w_1, \ldots, w_n \in W$, we have

$$c(w_1, \ldots, w_i, w_{i+1}, \ldots, w_n) = -c(w_1, \ldots, w_{i+1}, w_i, \ldots, w_n).$$

General elements w_i and w_j can change places in the entries of c by encountering the sign of the permutation exchanging the two entries. Observe that in case $\mathfrak{g}$ is of dimension n, $\mathrm{Alt}(\mathfrak{g}^n, V)$ has the same dimension as V and $\mathrm{Alt}(\mathfrak{g}^{n+1}, V) = 0$. This follows from the fact that there is a unique k-multilinear, alternating map $\mathrm{Alt}(\mathfrak{g}^n, k)$ up to scalar multiple in this case.

The Chevalley-Eilenberg complex is given by the following differential $d_p : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$. For $c \in C^p(\mathfrak{g}, V)$, we set

$$d_p c(x_1, \ldots, x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{p+1}) + \\ - \sum_{i=1}^{p+1} (-1)^i x_i \cdot c(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}).$$

Lemma 3.1. $d_{p+1} \circ d_p = 0$ *for all* $p \geq 0$.

Proof. One possible proof is by brute force computation. Another possible proof uses the simplicial structure of the map d_p. Here we present a proof relying on the Cartan calculus on the Chevalley-Eilenberg complex. Introduce the following operations for $x \in \mathfrak{g}$.

$$L_x : C^p(\mathfrak{g}, V) \to C^p(\mathfrak{g}, V),$$
$$c \mapsto ((x_1, \ldots, x_p) \mapsto x \cdot c(x_1, \ldots, x_p) - \sum c(x_1, \ldots, [x, x_i], \ldots, x_p).$$

and

$$i_x : C^p(\mathfrak{g}, V) \to C^{p-1}(\mathfrak{g}, V), \;\; c \mapsto ((x_1, \ldots, x_{p-1}) \mapsto c(x, x_1, \ldots, x_{p-1}).$$

We have the following relations between these operators and the Chevalley-Eilenberg differential d (where we now omit the dimensional index for better readability).

(a) $L_x = d \circ i_x + i_x \circ d$,

(b) $L_x \circ L_y - L_y \circ L_x = L_{[x,y]}$,

(c) $L_x \circ i_y - i_y \circ L_x = i_{[x,y]}$,

(d) $L_x \circ d = d \circ L_x$.

These are Cartan's formulas. They also hold on the de Rham complex of differential forms. Formulas (b) and (c) only hold for strictly positive p. We leave the proof of these formulas as Exercises.

Let us conclude the proof of the lemma by induction using (a)-(d). For $p = 0$, $x_1, x_2 \in \mathfrak{g}$ and $v \in V$, we have

$$\begin{aligned} d^2v(x_1, x_2) &= -dv([x_1, x_2]) + x_1 \cdot dv(x_2) - x_2 \cdot dv(x_1) \\ &= -[x_1, x_2] \cdot v + x_1 \cdot (x_2 \cdot v) - x_2 \cdot (x_1 \cdot v) = 0. \end{aligned}$$

For $p > 0$, $c \in C^p(\mathfrak{g}, V)$ and $x_1, \ldots, x_{p+2} \in \mathfrak{g}$, we have

$$\begin{aligned} (d^2c)(x_1, \ldots, x_{p+2}) &= (i_{x_1} \circ d \circ d)(c)(x_2, \ldots, x_{p+2}) \\ &= (L_{x_1} \circ d - d \circ i_{x_1} \circ d)(c)(x_2, \ldots, x_{p+2}) \\ &= (L_{x_1} \circ d - d \circ L_{x_1} + d \circ d \circ i_{x_1})(c)(x_2, \ldots, x_{p+2}) \\ &= 0. \end{aligned}$$

Here we have used (a) in the second and third line, and finally formula (d) and the induction hypothesis $d \circ d = 0$ on $C^{p-1}(\mathfrak{g}, V)$. □

Definition 3.2. The p-th comology space of the Lie algebra $\mathfrak{g}$ with values in the $\mathfrak{g}$-module V is by definition $H^p(C^*(\mathfrak{g}, V), d_*)$.

4 Interpretations of low degree cohomology spaces

In degree $p = 0$, the cohomology space $H^0(\mathfrak{g}, V)$ is simply the space of $\mathfrak{g}$-invariants $V^{\mathfrak{g}}$ in V:

$$V^{\mathfrak{g}} := \{v \in V \,|\, \forall x \in \mathfrak{g} : x \cdot v = 0\}.$$

This follows immediately from the facts that $H^0(\mathfrak{g}, V) = Z^0(\mathfrak{g}, V)$ and that the coboundary operator $d_0 : V \to C^1(\mathfrak{g}, V)$ is given by $v \mapsto (x \mapsto x \cdot v)$.

Example 4.1. (a) For the action of $\mathfrak{gl}_n(k)$ on k^n, the invariants are zero. Indeed, the zero vector $v = 0 \in k^n$ is the only vector which satisfies $x \cdot v = 0$ for all matrices $x \in \mathfrak{gl}_n(k)$.

(b) For the action of the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ of traceless antisymmetric matrices on $\mathbb{R}^3$, we have as invariants still only the zero vector.

In degree $p = 1$, we obtain

$$Z^1(\mathfrak{g}, V) = \{c : \mathfrak{g} \to V \mid \forall x, y \in \mathfrak{g} : c[x, y] - x \cdot c(y) + y \cdot c(x) = 0\}.$$

Observe that this is a derivation property for the linear map $c : \mathfrak{g} \to V$: c is called a derivation in case for all $x, y \in \mathfrak{g}$: $c[x, y] = x \cdot c(y) - y \cdot c(x)$. The space of derivations is denoted by $\mathfrak{der}\,(\mathfrak{g}, V)$. One calls *inner derivation* a derivation $c : \mathfrak{g} \to V$ which is obtained as $c_v(x) := x \cdot v$ for some $v \in V$. The space of inner derivations is denoted $\mathfrak{inder}\,(\mathfrak{g}, V)$. Thus trivially

$$H^1(\mathfrak{g}, V) = \mathfrak{der}\,(\mathfrak{g}, V) \,/\, \mathfrak{inder}\,(\mathfrak{g}, V).$$

For example, in the special case where $V = k$ is the trivial 1-dimensional $\mathfrak{g}$-module (i.e. all operations $x \cdot v = 0$), this gives

$$H^1(\mathfrak{g}, k) = \Big(\mathfrak{g} \,/\, [\mathfrak{g}, \mathfrak{g}]\Big)^*,$$

the k-linear dual of the quotient of $\mathfrak{g}$ with respect to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, i.e. the subspace generated by all commutators $[x, y]$ for $x, y \in \mathfrak{g}$. For the Lie algebra $\mathfrak{gl}_n(k)$, we obtain here

$$H^1(\mathfrak{gl}_n(k), k) = \Big(\mathfrak{gl}_n(k) \,/\, [\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)]\Big)^* = k^*,$$

because $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] = \mathfrak{sl}_n(k)$.

Proposition 4.2. *(a) A homomorphism of $\mathfrak{g}$-modules $f : V \to W$ induces a k-linear map*

$$f_* : H^*(\mathfrak{g}, V) \to H^*(\mathfrak{g}, W), \;\; [c] \mapsto [f \circ c].$$

(b) A short exact sequence of $\mathfrak{g}$-modules

$$0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$$

induces a long exact sequence in cohomology.

Proof. Both statements are exercises. Use Theorem 1.12 for the second assertion. □

5 Extensions and the second cohomology space

Definition 5.1. An abelian extension of Lie algebras is a short exact sequence of Lie algebras

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0,$$

where the Lie algebra $\mathfrak{a}$ is abelian.

Given an abelian extension, the Lie algebra $\mathfrak{h}$ acts on $\mathfrak{a}$ in the following way. For $x \in \mathfrak{h}$, there exists $\hat{x} \in \mathfrak{g}$ such that $\pi(\hat{x}) = x$. Let x act on $a \in \mathfrak{a}$ by $x \cdot a := [\hat{x}, i(a)]$. This is well-defined, as another lift $\hat{x}'$ of x is written $\hat{x}' = \hat{x} + a'$ for some $a' \in \mathfrak{a}$ and thus $x \cdot a = [\hat{x}, a] = [\hat{x}', a]$ because $\mathfrak{a}$ is abelian. The action property follows from the Jacobi identity. In the following, we will often suppress the map i on elements and consider elements of $\mathfrak{a}$ directly as elements of the extension.

In case this action of $\mathfrak{h}$ on $\mathfrak{a}$ defined in this way is trivial, one speaks of a *central extension*.

Definition 5.2. Two abelian extensions

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0$$

and

$$0 \to \mathfrak{a} \xrightarrow{i'} \mathfrak{g}' \xrightarrow{\pi'} \mathfrak{h} \to 0,$$

with the same kernel $\mathfrak{a}$ and the same cokernel $\mathfrak{h}$ are called equivalent if there exists a morphism of Lie algebras $\varphi : \mathfrak{g} \to \mathfrak{g}'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r, "i"] \arrow[d, "\mathrm{id}_{\mathfrak{a}}"] & \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\varphi"] & \mathfrak{h} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{h}}"] & 0 \\
0 \arrow[r] & \mathfrak{a} \arrow[r, "i'"] & \mathfrak{g}' \arrow[r, "\pi'"] & \mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

is commutative. Observe that in this case, the map φ is an isomorphism by the Five Lemma.

The upshot of this section is to show the following theorem:

Theorem 5.3. *The set of equivalence classes of abelian extensions with fixed cokernel $\mathfrak{h}$ and fixed kernel $\mathfrak{a}$ is in bijection with $H^2(\mathfrak{h}, \mathfrak{a})$.*

For this, we will first of all associate a 2-cocycle to an abelian extension. For a given abelian extension

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0,$$

choose a linear section $s : \mathfrak{h} \to \mathfrak{g}$ of π, i.e. $\pi \circ s = \mathrm{id}_{\mathfrak{h}}$. Such a section exists, because the sequence of vector spaces is split, while s cannot be taken to be a morphism of Lie algebras in general. We associate to s the failure to be a morphism of Lie algebras:

$$\alpha(x,y) := s[x,y] - [s(x), s(y)],$$

for all $x, y \in \mathfrak{h}$. As π is a morphism of Lie algebras, we have $\pi(\alpha(x,y)) = 0$, thus $\alpha(x,y)$ has values in $\ker(\pi) = \mathrm{im}\,(i) = \mathfrak{a}$. We claim that $\alpha : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ is a 2-cocycle. Indeed,

$$\begin{aligned}
&-\alpha([x,y],z) - \alpha([y,z],x) - \alpha([z,x],y) + x\cdot\alpha(y,z) + y\cdot\alpha(z,x) + z\cdot\alpha(x,y) = \\
&= -s[[x,y],z] + [s[x,y],s(z)] - s[[y,z],x] + [s[y,z],s(x)] - s[[z,x],y] + [s[z,x],s(y)] + \\
&\quad + [s(x),s[y,z]] - [s(x),[s(y),s(z)]] + [s(y),s[z,x]] - [s(y),[s(z),s(x)]] + \\
&\quad + [s(z),s[x,y]] - [s(z),[s(x),s(y)]] \\
&= [s[x,y],s(z)] + [s[y,z],s(x)] + [s[z,x],s(y)] + [s(x),s[y,z]] + [s(y),s[z,x]] + [s(z),s[x,y]] = 0
\end{aligned}$$

In fact, one can show that the cohomology class $[\alpha] \in H^2(\mathfrak{g},\mathfrak{a})$ does not depend on the choice of the section s and coincides for two equivalent abelian extensions.

In order to show the surjectivity of the map sending an equivalence class of abelian extensions to the corresponding class $[\alpha] \in H^2(\mathfrak{g},\mathfrak{a})$, we have to construct an extension from a given cohomology class $[\alpha]$. This goes as follows.

Define the following bracket on the vector space $\mathfrak{g} := \mathfrak{a} \oplus \mathfrak{h}$.

$$[(a,x),(b,y)] := (x\cdot b - y\cdot a + \alpha(x,y), [x,y]).$$

We claim that this bracket renders $\mathfrak{g}$ a Lie algebra: The Jacobi identity follows from the Jacobi identity in $\mathfrak{h}$ and the cocycle identity of α. It is clear that the 2-cocycle associated to this extension is the α we started with.

Example 5.4. (a) A very basic central extension is the central extension of every Lie algebra $\mathfrak{g}$ by its center $Z(\mathfrak{g})$. For example, the center of $\mathfrak{gl}_n(k)$ is $Z(\mathfrak{gl}_n(k)) = k$, given by the homotheties. This central extension is split, i.e. its class in $H^2(\mathfrak{sl}_n(k),k)$ is zero (for any field k !). In fact, $H^2(\mathfrak{sl}_n(k),k) = 0$ by Whitehead's Lemma for k of characteristic zero.

(b) Central extensions are important in Quantum mechanics, because wave functions are determined only up to a phase factor which may be interpreted as giving rise to a central extension of the structure group/Lie algebra. For example, a famous central extension arising in physics is the *Virasoro algebra*, the 1-dimensional central extension of the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle. $\mathrm{Vect}(S^1)$ has a basis $e_n := x^{n+1}\frac{d}{dx}$ for $n \in \mathbb{Z}$ with relations $[e_n, e_m] = (m-n)e_{n+m}$, and the cocycle specifying the central extension is

$$\alpha(e_n, e_m) := \frac{\delta_{m+n,0}}{12}(n^3 - n).$$

The bracket for the Virasoro algebra $\mathcal{V}ir = \mathbb{C} \oplus \mathrm{Vect}(S^1)$ reads

$$[(a,x),(b,y)] = (\alpha(x,y),[x,y]).$$

(c) The trivial abelian extension (i.e. the abelian extension representing the zero class in $H^2(\mathfrak{g}, \mathfrak{a})$) is simply the semidirect product, i.e. the bracket

$$[(a, x), (b, y)] := (x \cdot b - y \cdot a, [x, y])$$

on the vector space $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$.

(d) Another central extension which is important in physics is the *Heisenberg Lie algebra*. This is the central extension of the abelian Lie algebra $\mathbb{R}^2$ generated by the basis $x, y \in \mathbb{R}^2$, extended by the cocycle

$$\alpha(x, y) = -\alpha(y, x) = z,$$

where z is the central element. Abelian Lie algebras always have lots of cohomology and the class of α is non-zero in $H^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^2$.

6 Crossed Modules of Lie algebras

In the same way as abelian extensions represent 2-cohomology, we will see later on that crossed modules of Lie algebras represent 3-cohomology.

Definition 6.1. A crossed module of Lie algebras is the data of a homomorphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action η of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations, denoted $\eta : \mathfrak{n} \to \mathfrak{Der}(\mathfrak{m})$ or sometimes simply $m \mapsto n \cdot m$ for all $m \in \mathfrak{m}$ and all $n \in \mathfrak{n}$, such that

(a) $\mu(n \cdot m) = [n, \mu(m)]$ for all $n \in \mathfrak{n}$ and all $m \in \mathfrak{m}$,

(b) $\mu(m) \cdot m' = [m, m']$ for all $m, m' \in \mathfrak{m}$.

Remark 6.2. Property (a) means that the morphism μ is equivariant with respect to the $\mathfrak{n}$-action via η on $\mathfrak{m}$ and the adjoint action on $\mathfrak{n}$. Property (b) is called *Peiffer identity.*

Remark 6.3. To each crossed module of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$, one associates a four term exact sequence

$$0 \to V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \to 0$$

where $\ker(\mu) =: V$ and $\mathfrak{g} := \operatorname{coker}(\mu)$.

Remark 6.4. (a) By property (a), $\operatorname{im}(\mu)$ is an ideal, and thus $\mathfrak{g}$ is a Lie algebra.

(b) By property (b), V is a central ideal of $\mathfrak{m}$, and in particular abelian.

(c) Lifting elements of $\mathfrak{g}$ to $\mathfrak{n}$, the action of $\mathfrak{n}$ on $\mathfrak{m}$ induces an *outer action* of $\mathfrak{g}$ on $\mathfrak{m}$. Given linear sections ρ and ρ' of π and an element $x \in \mathfrak{g}$, $\eta(\rho(x))$ and $\eta(\rho'(x))$ differ by the inner derivation $\mathrm{ad}_{m'}$ for some $m' \in \mathfrak{m}$. Indeed,

$$\begin{aligned} \eta(\rho(x)) - \eta(\rho'(x)) &= \eta((\rho - \rho')(x))(m) \\ &= \eta(\mu(m'))(m) \\ &= [m', m]. \end{aligned}$$

Here we used property (b) in the last line, and m' exists by exactness of the four term sequence, because $(\rho - \rho')(x) \in \ker(\pi)$. This means that the expression $\eta(\rho(x))$ is well-defined up to inner isomorphism.

Moreover, $\eta \circ \rho$ satisfies the requirements of an action also up to inner derivations. Indeed, denote for all $x, y \in \mathfrak{g}$ by

$$\alpha(x, y) := [\rho(x), \rho(y)] - \rho([x, y]),$$

the default of ρ to be a morphism of Lie algebras. Then $\alpha(x, y) \in \ker(\pi)$, because π is a morphism and ρ a section of π. Therefore by exactness of the four term sequence, there exists $\beta(x, y) \in \mathfrak{m}$ such that

$$\mu(\beta(x, y)) = \alpha(x, y).$$

Then to show that $\eta \circ \rho$ is an action, we have to consider $\eta([\rho(x), \rho(y)] - \rho([x, y]))(m)$ for some $m \in \mathfrak{m}$. But this gives

$$\eta(\alpha(x, y))(m) = \eta(\mu(\beta(x, y)))(m) = [\beta(x, y), m]$$

by property (b), and in this sense, an outer action is an action up to inner derivations.

Now property (a) implies that the restriction of this outer action to V induces the structure of a $\mathfrak{g}$-module on V.

(d) Note that by this lifting procedure, the action η (resp. the adjoint action) does not in general render $\mathfrak{m}$ (resp. $\mathfrak{n}$) a $\mathfrak{g}$-module.

Example 6.5. Let us list some elementary examples of crossed modules:

(a) Each central extension is a crossed module. Indeed, central extensions correspond exactly to the case where the map μ is surjective.

(b) Each (inclusion of an) ideal in a Lie algebra constitutes a crossed module. Indeed, an (inclusion of an) ideal correspond exactly to the case where the map μ is injective.

(c) For each Lie algebra $\mathfrak{l}$, there is a canonical crossed module

$$\mu : \mathfrak{l} \to \mathfrak{der}(\mathfrak{l}),$$

where μ sends an element $x \in \mathfrak{l}$ to the inner derivation ad_x defined by $\mathrm{ad}_x(y) = [x, y]$ for all $y \in \mathfrak{l}$. The action of $\mathfrak{der}(\mathfrak{l})$ on $\mathfrak{l}$ is the usual action as a derivation. The kernel of μ is the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ and the cokernel of μ is the Lie algebra of outer derivations

$$\mathfrak{out}(\mathfrak{l}) := \mathfrak{der}(\mathfrak{l})/\mathrm{ad}(\mathfrak{l}).$$

Definition 6.6. Two crossed modules $\mu : \mathfrak{m} \to \mathfrak{n}$ (with action η) and $\mu' : \mathfrak{m}' \to \mathfrak{n}'$ (with action η') such that $\ker(\mu) = \ker(\mu') =: V$ and $\operatorname{coker}(\mu) = \operatorname{coker}(\mu') =: \mathfrak{g}$ are called elementary equivalent if there are morphisms of Lie algebras $\varphi : \mathfrak{m} \to \mathfrak{m}'$ and $\psi : \mathfrak{n} \to \mathfrak{n}'$ which are compatible with the actions, meaning

$$\varphi(\eta(n)(m)) = \eta'(\psi(n))(\varphi(m)),$$

for all $n \in \mathfrak{n}$ and all $m \in \mathfrak{m}$, and such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "\mathrm{id}_V"] & \mathfrak{m} \arrow[r, "\mu"] \arrow[d, "\varphi"] & \mathfrak{n} \arrow[r, "\pi"] \arrow[d, "\psi"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r, "i'"] & \mathfrak{m}' \arrow[r, "\mu'"] & \mathfrak{n}' \arrow[r, "\pi'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

We call equivalence of crossed modules the equivalence relation generated by elementary equivalence. One easily sees that two crossed modules are equivalent in case there exists a zig-zag of elementary equivalences going from one to the other (which are not necessarily going all in the same direction).

Let us denote by $\mathrm{crmod}(\mathfrak{g}, V)$ the set of equivalence classes of Lie algebra crossed modules with respect to fixed kernel V and fixed cokernel $\mathfrak{g}$.

Remark 6.7. Compare this equivalence relation to the equivalence of two abelian extensions, cf Definition 5.2. In the framework of extensions, equivalence imposes the underlying vector space of the extension up to isomorphism. For crossed modules, this is not the case, and leads thus to much more different representatives of the equivalence class of a crossed module.

We will now define the *sum of two crossed modules*. Consider two crossed modules $\mu : \mathfrak{m} \to \mathfrak{n}$ and $\mu' : \mathfrak{m}' \to \mathfrak{n}'$ with isomorphic kernel and cokernel and their corresponding four term exact sequences

$$0 \to V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \to 0$$

and

$$0 \to V \xrightarrow{i'} \mathfrak{m}' \xrightarrow{\mu'} \mathfrak{n}' \xrightarrow{\pi'} \mathfrak{g} \to 0.$$

Denote by $K := \{(v, -v) \in V \oplus V\}$ the kernel of the addition map $V \oplus V \to V$. Notice that the diagonal $\triangle : V \to V \oplus V$ followed by the quotient map $V \oplus V \to (V \oplus V)/K$ identifies V and $(V \oplus V)/K$. K can be considered as a subspace in $\mathfrak{m} \oplus \mathfrak{m}'$ via $i \oplus i'$. As V is central in $\mathfrak{m}$ and $\mathfrak{m}'$, K is an ideal of $\mathfrak{m} \oplus \mathfrak{m}'$.

Denote by $\mathfrak{n} \oplus_{\mathfrak{g}} \mathfrak{n}'$ the pullback associated to the maps $\pi : \mathfrak{n} \to \mathfrak{g}$ and $\pi' : \mathfrak{n}' \to \mathfrak{g}$. More explicitly,

$$\mathfrak{n} \oplus_{\mathfrak{g}} \mathfrak{n}' = \{(n, n') \in \mathfrak{n} \oplus \mathfrak{n}' \mid \pi(n) = \pi'(n')\}.$$

Notice that the two maps $\pi : (\mathfrak{n} \oplus_{\mathfrak{g}} \mathfrak{n}') \to \mathfrak{g}$ and $\frac{1}{2}(\pi + \pi') : (\mathfrak{n} \oplus_{\mathfrak{g}} \mathfrak{n}') \to \mathfrak{g}$ coincide (because in our base field, 2 is invertible).

With these preparations, we have the following definition:

Definition 6.8. The sum of two crossed modules $\mu : \mathfrak{m} \to \mathfrak{n}$ and $\mu' : \mathfrak{m}' \to \mathfrak{n}'$ such that $\ker(\mu) = \ker(\mu') = V$ and $\mathrm{coker}(\mu) = \mathrm{coker}(\mu) = \mathfrak{g}$ is by definition the crossed module

$$0 \to V \stackrel{(i \oplus i')\circ\triangle}{\longrightarrow} (\mathfrak{m} \oplus \mathfrak{m}')/K \stackrel{\mu\oplus\mu'}{\to} (\mathfrak{n} \oplus_{\mathfrak{g}} \mathfrak{n}') \stackrel{\pi}{\to} \mathfrak{g} \to 0$$

The action of $\mathfrak{n} \oplus_{\mathfrak{g}} \mathfrak{n}'$ on $(\mathfrak{m} \oplus \mathfrak{m}')/K$ by derivations is induced from the sum of actions on the two summands. The compatibility relations (a) and (b) of Definition 6.1 are true in the direct sum, thus true for the crossed module sum.

Lemma 6.9. *The sum of crossed modules defines an abelian group structure on the set of equivalence classes of crossed modules* $\mathrm{crmod}(\mathfrak{g}, V)$ *with given kernel* V *and cokernel* $\mathfrak{g}$.

Proof. It is clear that the sum of crossed modules is associative and commutative as it is induced by the direct sum. It is equally clear that the sum is compatible with the equivalence relation as we can sum the maps giving the equivalences.

We have to show that there is a zero element and an inverse to every crossed module. We define the zero crossed module with given kernel V and cokernel $\mathfrak{g}$ to be

$$0 \to V \stackrel{\mathrm{id}_V}{\to} V \stackrel{0}{\to} \mathfrak{g} \stackrel{\mathrm{id}_{\mathfrak{g}}}{\to} \mathfrak{g} \to 0$$

and the inverse of a crossed module

$$0 \to V \stackrel{i}{\to} \mathfrak{m} \to \mathfrak{n} \to \mathfrak{g} \to 0$$

to be

$$0 \to V \stackrel{-i}{\to} \mathfrak{m} \to \mathfrak{n} \to \mathfrak{g} \to 0.$$

In order to show that this crossed module is inverse to the given one, notice that $\mathrm{crmod}(\mathfrak{g}, -)$ is an additive functor. Thus we have for an equivalence class $[\mu : \mathfrak{m} \to \mathfrak{n}] \in \mathrm{crmod}(\mathfrak{g}, V)$

$$(\alpha_1 + \alpha_2)[\mu : \mathfrak{m} \to \mathfrak{n}] = \alpha_1[\mu : \mathfrak{m} \to \mathfrak{n}] + \alpha_2[\mu : \mathfrak{m} \to \mathfrak{n}],$$

where $\alpha_i : V \to V'$ $(i = 1, 2)$ are two $\mathfrak{g}$-module morphisms. This reduces the proof to showing that pushforward by the zero map $0 : V \to V$ gives the zero class. Now, pushforward by the zero map splits up a direct factor V in $(V \oplus \mathfrak{m})/(0 \oplus (-i(V)))$ and we have then a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "0"] & \mathfrak{m} \arrow[r, "\mu"] \arrow[d, "\mathrm{incl}_2"] & \mathfrak{n} \arrow[r, "\pi"] \arrow[d, "\mathrm{id}_{\mathfrak{n}}"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[d, "\mathrm{id}_V"] & (V \oplus \mathfrak{m})/(0 \oplus (-i(V))) \arrow[r, "0\oplus\mu"] \arrow[d, "\mathrm{proj}_1"] & \mathfrak{n} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r, "\mathrm{id}_V"] & V \arrow[r, "0"] & \mathfrak{g} \arrow[r, "\mathrm{id}_{\mathfrak{g}}"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Here incl_2 and proj_2 are the standard inclusions and projections to/from the direct sum. This shows that the class $0[\mu : \mathfrak{m} \to \mathfrak{n}]$ is the zero class. $\square$

Crossed modules may serve as explicit representatives of third cohomology classes, this is the essence of the following theorem, which Mac Lane attributes to Gerstenhaber:

Theorem 6.10. *There is an isomorphism of abelian groups*

$$b : \mathrm{crmod}(\mathfrak{g}, V) \cong H^3(\mathfrak{g}, V).$$

In a first step, we will only discuss in this section how to associate a cohomology class in $H^3(\mathfrak{g}, V)$ to a given crossed module.

Let us show how to associate to a crossed module a 3-cocycle of $\mathfrak{g}$ with values in V. For this, recall the exact sequence from Remark 6.3:

$$0 \to V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \xrightarrow{\pi} \mathfrak{g} \to 0$$

The first step is to take a linear section ρ of π and to compute the failure of ρ to be a Lie algebra homomorphism, i.e.

$$\alpha(x_1, x_2) = [\rho(x_1), \rho(x_2)] - \rho([x_1, x_2]).$$

Here, $x_1, x_2 \in \mathfrak{g}$. α is bilinear and skewsymmetric in x_1, x_2. We have obviously $\pi(\alpha(x_1, x_2)) = 0$, because π is a Lie algebra homomorphism, so $\alpha(x_1, x_2) \in \mathrm{im}\,(\mu) = \ker\,(\pi)$. This means by exactness that there exists $\beta(x_1, x_2) \in \mathfrak{m}$ such that

$$\mu(\beta(x_1, x_2)) = \alpha(x_1, x_2).$$

Choosing a linear section σ on $\mathrm{im}\,(\mu)$, one can choose β as

$$\beta(x_1, x_2) = \sigma(\alpha(x_1, x_2)) \tag{1}$$

showing that we can suppose β bilinear and skewsymmetric in x_1, x_2.

Now, $\mu(d\beta(x_1, x_2, x_3)) = 0$ by the following lemma:

Lemma 6.11. $\mu(d\beta(x_1, x_2, x_3)) = 0$ *where d is the formal expression of the Lie algebra cohomology boundary operator corresponding to cohomology of $\mathfrak{g}$ with values in $\mathfrak{m}$, and $x_1, x_2, x_3 \in \mathfrak{g}$.*

Proof.

$$\begin{aligned}
\mu(d\beta(x_1,x_2,x_3)) &= \mu\Big(\sum_{\text{cycl.}}\beta([x_1,x_2],x_3) - \sum_{\text{cycl.}}\eta(\rho(x_1))\cdot\beta(x_2,x_3)\Big) \\
&= \sum_{\text{cycl.}}\alpha([x_1,x_2],x_3) - \sum_{\text{cycl.}}[\rho(x_1),\mu(\beta(x_2,x_3))] \\
&= \sum_{\text{cycl.}}\alpha([x_1,x_2],x_3) - \sum_{\text{cycl.}}[\rho(x_1),\alpha(x_2,x_3)] \\
&= \sum_{\text{cycl.}}\Big([\rho([x_1,x_2]),\rho(x_3)] - \rho([[x_1,x_2],x_3])]\Big) \\
&- \sum_{\text{cycl.}}\Big([\rho(x_1),[\rho(x_2),\rho(x_3)]] + [\rho(x_1),\rho([x_2,x_3])]\Big) = 0.
\end{aligned}$$

□

This means that $d\beta(x_1,x_2,x_3) \in \ker(\mu) = \operatorname{im}(i) = i(V)$, i.e. there exists $\gamma(x_1,x_2,x_3) \in V$ such that $d\beta(x_1,x_2,x_3) = i(\gamma(x_1,x_2,x_3))$. The explicit formula of $d\beta(x_1,x_2,x_3)$ shows trilinearity and skewsymmetry in the three variables x_1, x_2 and x_3. Choosing a linear section τ on $i(V) = \ker(\mu)$, one can choose γ to be $\tau \circ d\beta$ (in the obvious sense) gaining that γ is also trilinear and skewsymmetric in x_1, x_2 and x_3. By the following lemma, γ is a 3-cocycle of $\mathfrak{g}$ with values in V:

Lemma 6.12. *γ is a 3-cocycle of $\mathfrak{g}$ with values in V.*

Proof. We have to show that $d\gamma(x_1,x_2,x_3,x_4) = 0$, denoting by x_1,x_2,x_3,x_4 four elements of $\mathfrak{g}$ and by d the Lie algebra coboundary operator of $\mathfrak{g}$ with values in V. The expression for $d\gamma(x_1,x_2,x_3,x_4)$ is the sum of "action terms" and "bracket terms". It is enough to show $i(d\gamma(x_1,x_2,x_3,x_4)) = 0$.

$$\begin{aligned}
i(d\gamma(x_1,x_2,x_3,x_4)) &= di(\gamma(x_1,x_2,x_3,x_4)) \\
&= d\circ d\beta(x_1,x_2,x_3,x_4).
\end{aligned}$$

One has to be careful because $d\circ d$ is not automatically zero, because of the fact that $\eta\circ\rho$ is not an action of $\mathfrak{g}$ on $\mathfrak{m}$ in general. Let us display here only some terms of it, while the other terms vanish as usual. The terms we choose are all "action terms of the action terms" and some "action terms of the bracket terms".

$$\begin{aligned}
i(d\gamma(x_1,x_2,x_3,x_4)) &= \sum_{1\leq i<j\leq 4}(-1)^{i+j}\eta(\rho([x_i,x_j]))\beta(x_1,\ldots,\hat{x_i},\ldots,\hat{x_j},\ldots,x_4) \\
&- \sum_{i=1}^{4}(-1)^i\eta(\rho(x_i))\sum_{l=1}^{3}(-1)^l\eta(\rho(z_l))\beta(z_1,\ldots,\hat{z_l},\ldots,z_3).
\end{aligned}$$

Here, we denote by z_1, z_2, z_3 the three remaining x_r after having chosen x_i from the list. Now, the difference of acting by $\eta(\rho([x, y]))$ and acting by $\eta(\rho(x))\eta(\rho(y)) - \eta(\rho(y))\eta(\rho(x))$ is just the action by $\eta(\alpha(x, y))$. Calculating the differences of the actions by the bracket and the action of the single elements gives thus terms of the form

$$\eta(\alpha(x, y))\beta(u, v) = \eta(\mu(\beta(x, y)))\beta(u, v) = [\beta(x, y), \beta(u, v)],$$

where the last equality follows from property (b) in Definition 6.1. Then the sum reads

$$\begin{array}{l} [\beta(x_1, x_2), \beta(x_3, x_4)] - [\beta(x_2, x_3), \beta(x_4, x_1)] + [\beta(x_3, x_4), \beta(x_1, x_2)] \\ -[\beta(x_4, x_1), \beta(x_2, x_3)] + [\beta(x_4, x_2), \beta(x_1, x_3)] - [\beta(x_1, x_3), \beta(x_2, x_4)] \end{array}$$

and vanishes. □

Lemma 6.13. *The class of the cocycle γ in $C^3(\mathfrak{g}, V)$ does not depend on the choice of the sections ρ, σ and τ.*

Proof. (1) Let ρ and ρ' be two sections of π. Denote by $\alpha(x, y)$ resp. $\alpha'(x, y)$, $\beta(x, y)$ resp. $\beta'(x, y)$, $\gamma(x, y, z)$ resp. $\gamma'(x, y, z)$ the elements of $\mathfrak{n}$, $\mathfrak{m}$ and V constructed above with respect to ρ and ρ'. Here $x, y, z \in \mathfrak{g}$. By construction, we have

$$\rho'(x) = \rho(x) + \delta(x),$$

for some linear map $\delta : \mathfrak{g} \to \ker(\pi) \subset \mathfrak{n}$. But then α' may be written

$$\begin{array}{rcl} \alpha'(x, y) & = & [\rho'(x), \rho'(y)] - \rho'([x, y]) \\ & = & [(\rho + \delta)(x), (\rho + \delta)(y)] - (\rho + \delta)([x, y]) \\ & = & \alpha(x, y) + [\rho(x), \delta(y)] + [\delta(x), \rho(y)] + [\delta(x), \delta(y)] - \delta([x, y]). \end{array}$$

Observe that the expression $[\rho(x), \delta(y)] + [\delta(x), \rho(y)] - \delta([x, y])$ is just the formal coboundary $d\delta(x, y)$, where $\delta : \mathfrak{g} \to \mathfrak{n}$ is considered as a cochain with values in $\mathfrak{n}$ although lifting elements, $\mathfrak{n}$ is in general not a $\mathfrak{g}$-module via the adjoint action.

As $d\delta(x, y)$ lies in the kernel of π, there exists $\epsilon(x, y) \in \mathfrak{m}$ such that $\mu \circ \epsilon = d\delta$, and as before, we may take ϵ bilinear. In the same way, as $[\delta(x), \delta(y)]$ is in $\ker(\epsilon)$, there exists $\theta(x, y) \in \mathfrak{m}$ with $\mu\theta(x, y) = [\delta(x), \delta(y)]$. Therefore we get

$$\mu\beta'(x, y) = \mu\beta(x, y) + \mu\epsilon(x, y) + \mu\theta(x, y),$$

and hence there exists an element in $\ker(\mu) = \mathrm{im}(i)$, denoted $i(\zeta)$, such that

$$\beta'(x, y) = \beta(x, y) + \epsilon(x, y) + \theta(x, y) + i(\zeta)(x, y).$$

When applying in the next step d to all terms, the ζ will give a coboundary, because $d(i(\zeta)) = i(d^V\zeta)$. Let us treat the term $\epsilon(x, y)$. Observe that using the linear section σ on $\mathrm{im}\,(\mu)$, we have $\epsilon = \sigma d\delta$. Actually, we have

$$\begin{aligned}\mu\Big(\sigma d\delta(x,y) - d\sigma\delta(x,y)\Big) &= \mu\sigma\Big(\rho(x)\cdot\delta(y) - \rho(y)\cdot\delta(x) + \delta([x,y])\Big) \\ &- \mu\Big(\eta(\rho(x))(\sigma\delta(y)) + \eta(\rho(y))(\sigma\delta(x)) - \sigma\delta([x,y])\Big) \\ &= 0,\end{aligned}$$

using property (a) of a crossed module. This means that the difference is in the kernel of μ, thus in the image of i, and replacing ϵ by $d\sigma\delta$ adds only another coboundary in the end. A similar reasoning applies to the term $\theta(x,y)$. In conclusion, we have shown that changing the section ρ results in changing the cocycle γ by a coboundary.

(2) Now suppose that we chose two different linear sections σ and σ' of μ on $\mathrm{im}\,(\mu)$, leading to different lifts $\beta = \sigma\alpha$ and $\beta' = \sigma'\alpha$. We have then $\beta - \beta' \in \ker(\mu) = \mathrm{im}\,(i)$, and we have already seen that this leads to the corresponding γ and γ' differing by a coboundary.

(3) Two sections τ and τ' of i have to be the same; they are both inverses of the isomorphism i on its image. □

Lemma 6.14. *Let $\mu : \mathfrak{m} \to \mathfrak{n}$ (with action η) and $\mu' : \mathfrak{m}' \to \mathfrak{n}'$ (with action η') such that $\ker(\mu) = \ker(\mu') =: V$ and $\mathrm{coker}\,(\mu) = \mathrm{coker}\,(\mu') =: \mathfrak{g}$ be two elementary equivalent crossed modules.*

Then the corresponding cohomology classes $b([\mu]) = [\gamma]$ and $b([\mu']) = [\gamma']$ coincide in $H^3(\mathfrak{g}, V)$.

Proof. Denote by (φ, ψ) the morphism rendering the two crossed modules elementary equivalent, see Definition 6.6. Denote further by γ (resp. γ') the cocycle associated to $\mu : \mathfrak{m} \to \mathfrak{n}$ (resp. to $\mu' : \mathfrak{m}' \to \mathfrak{n}'$) by choosing sections ρ, σ and τ (resp. ρ', σ' and τ') as in the preceding Lemma.
ρ being a section of π, $\tilde{\rho}' := \psi \circ \rho$ is a section of π'. Lemma 6.13 shows that α' (defined by the section ρ') and $\tilde{\alpha}' := \psi \circ \alpha$ (defined by the section $\tilde{\rho}'$) give rise to the same cohomology class $[\gamma']$ represented by γ'.

Now let us compute

$$(\tilde{d}(\sigma' \circ \psi \circ \alpha) - \tilde{d}(\varphi \circ \beta))(x,y,z)$$

for $x, y, z \in \mathfrak{g}$ where $\tilde{d}$ is the formal Lie algebra coboundary operator with values in $\mathfrak{m}'$ and with the formal action $\eta' \circ \psi \circ \rho$. Let us call $\tilde{\beta}' := \sigma' \circ \psi \circ \alpha$.

First we remark that $(\tilde{\beta}' - \varphi\circ\beta)(x,y) \in \ker(\mu')$, thus we introduce $v(x,y) \in V$ such that $(\tilde{\beta}' - \varphi \circ \beta)(x,y) = i'v(x,y)$ for all $x, y \in \mathfrak{g}$. A computation using that (φ, ψ) satisfies the conditions of Definition 6.6 shows that

$$\tilde{d}(\varphi \circ \beta)(x,y,z) \;=\; \varphi(d\beta(x,y,z)).$$

Another computation shows that $(\tilde{d}i'v)(x,y,z) = i'(dv(x,y,z))$. We thus obtain

$$(\gamma' - \gamma)(x,y,z) = dv(x,y,z),$$

showing that the two elementary equivalent crossed modules have the same cohomology class. □

In conclusion, we have constructed a well-defined map b associating to an equivalence class of crossed modules with kernel V and cokernel $\mathfrak{g}$ a cohomology class in $H^3(\mathfrak{g}, V)$. Theorem 6.10 states that this map is an isomorphism of abelian groups.

7 Construction of crossed modules

In order to motivate the construction procedure which we present below let us go back to the four term exact sequence associated to a crossed module:

$$0 \to V \stackrel{i}{\to} \mathfrak{m} \stackrel{\mu}{\to} \mathfrak{n} \stackrel{\pi}{\to} \mathfrak{g} \to 0$$

We have already seen that this is an exact sequence of Lie algebras. We derive from it now two short exact sequences of Lie algebras:

$$0 \to \mathfrak{m}/i(V) \stackrel{\mu}{\to} \mathfrak{n} \stackrel{\pi}{\to} \mathfrak{g} \to 0,$$

and

$$0 \to V \stackrel{i}{\to} \mathfrak{m} \stackrel{\mu}{\to} \mathrm{im}\,(\mu) \to 0.$$

Obviously, the crossed module $\mu : \mathfrak{m} \to \mathfrak{n}$ can be seen as the gluing of these two short exact sequences. The second sequence is a central extension, whereas the first is not in general, not even an abelian extension in general. We call this a *general extension* of Lie algebras.

Let us ask (some version of) the converse question: Given a Lie algebra $\mathfrak{g}$, a short exact sequence of $\mathfrak{g}$-modules

$$0 \to V_1 \to V_2 \to V_3 \to 0 \tag{2}$$

(regarded as a short exact sequence of abelian Lie algebras) and an abelian extension $\mathfrak{e}$ of $\mathfrak{g}$ by the abelian Lie algebra V_3

$$0 \to V_3 \to \mathfrak{e} \to \mathfrak{g} \to 0, \tag{3}$$

is the Yoneda product of the sequences in Equations (2) and (3) a crossed module ? In case the sequence (3) is given by a 2-cocycle α, we continue to use the notation $\mathfrak{e} = V_3 \times_\alpha \mathfrak{g}$.

Theorem 7.1. *In the above situation, the gluing of sequences* (2) *and* (3) *is a crossed module, the associated* 3*-cocycle of which is the image of the* 2*-cocycle defining the central extension* (3) *under the connecting homomorphism in the long exact cohomology sequence associated to the coefficient sequence* (2)*.*

Proof. Gluing the sequences in Equations (2) and (3) together, one obtains a map

$$\mu : V_2 \to \mathfrak{e}. \tag{4}$$

Writing $\mathfrak{e} = V_2 \oplus \mathfrak{g}$ as vector spaces, we have $\mu(v) = (v, 0)$. On the other hand, the $\mathfrak{e}$-action η on V_2 is induced by the action of $\mathfrak{g}$ on V_2: $\eta(w, x)(v) := x \cdot v$, where $(w, x) \in \mathfrak{e}$, $v \in V_2$. With these structures, condition (b) for a Lie algebra crossed module is trivially true, while condition (a) is true by definition of the bracket in the abelian extension: $\mu(\eta(w, x)(v)) = (x \cdot v, 0) = [(w, x), (v, 0)]$.

Now let us discuss the second claim. The short exact sequence (2) induces a short exact sequence of complexes

$$0 \to C^*(\mathfrak{g}, V_1) \stackrel{i}{\to} C^*(\mathfrak{g}, V_2) \stackrel{\pi}{\to} C^*(\mathfrak{g}, V_3) \to 0.$$

Take a cocycle $\alpha \in C^2(\mathfrak{g}, V_3)$, then the connecting homomorphism $\partial : C^2(\mathfrak{g}, V_3) \to C^3(\mathfrak{g}, V_1)$ is defined as follows:

$$\begin{tikzcd}
 & \beta \in \pi^{-1}(\alpha) \arrow[r, mapsto, "\pi"] \arrow[d, dash] & \alpha \arrow[d, dash] \\
C^2(\mathfrak{g}, V_1) \arrow[r, tail, "i"] \arrow[d, "d^{V_1}"] & C^2(\mathfrak{g}, V_2) \arrow[r, two heads, "\pi"] \arrow[d, dash, "d^{V_2}"] & C^2(\mathfrak{g}, V_3) \arrow[d, dash, "d^{V_3}"] \\
C^3(\mathfrak{g}, V_1) \arrow[r, tail, "i"] & C^3(\mathfrak{g}, V_2) \arrow[r, two heads, "\pi"] \arrow[d] & C^3(\mathfrak{g}, V_3) \arrow[d] \\
\partial\alpha := i^{-1}(d^{V_2}\beta) \arrow[r, mapsto, "i"] & d^{V_2}\beta \arrow[r, mapsto, "\pi"] & 0 = d^{V_3}\alpha
\end{tikzcd}$$

Here we wrote elements on top resp. on bottom of the corresponding spaces, and denoted by d^{V_1}, d^{V_2} and d^{V_3} the Lie algebra coboundaries with values in V_1, V_2 and V_3 respectively. Summarizing, $\partial\alpha$ is constructed by choosing an element β preimage of α under π, taking $d^{V_2}\beta$ and taking an preimage of $d^{V_2}\beta$ under i. It is obvious that this is exactly how we constructed the 3-cocycle γ corresponding to a crossed module. Having stated the coincidence of the two constructions, it remains to take for $\alpha \in C^2(\mathfrak{g}, V_3)$ the cocycle defining the abelian extension (3). $\square$

Now we can end the proof of Theorem 6.10 using the previous construction.
Surjectivity of the map b from Theorem 6.10:

We have to show that given a cohomology class $[\gamma] \in H^3(\mathfrak{g}, V)$, there is a crossed module whose associated class is $[\gamma]$. V is here some $\mathfrak{g}$-module. As the category of $\mathfrak{g}$-modules posesses enough injectives, there is an injective $\mathfrak{g}$-module I and a monomorphism $i : V \hookrightarrow I$. Consider now the short exact sequence of $\mathfrak{g}$-modules:

$$0 \to V \to I \to Q \to 0$$

where Q is the cokernel of i. As I is injective, the long exact sequence in cohomology gives $H^2(\mathfrak{g}, Q) \cong H^3(\mathfrak{g}, V)$. Thus $[\gamma]$ corresponds under the isomorphism (which is induced by the connecting homomorphism) to a class $[\alpha] \in H^2(\mathfrak{g}, Q)$, and the principal construction applied to the above short exact sequence of $\mathfrak{g}$-modules and the abelian extension of $\mathfrak{g}$ by Q using the cocycle α gives a crossed module whose class is a preimage under b of $[\gamma]$.

Injectivity of the map b from Theorem 6.10:

It is clear that b is a homomorphism of abelian groups. Thus in order to show the injectivity of b, it suffices to show that its kernel is trivial.

(a) A crossed module

$$0 \to V \xrightarrow{i} \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \to \mathfrak{g} \to 0$$

corresponding to the trivial cohomology class gives rise to a general extension

$$0 \to \mathfrak{m} \to \mathfrak{e} \to \mathfrak{g} \to 0.$$

Indeed, if $[\theta] = 0$, there exists $\omega \in C^2(\mathfrak{g}, V)$ such that $d\omega = \theta$. Set $\zeta := \beta(x,y) - i(\omega(x,y))$ for $x, y \in \mathfrak{g}$. Then

$$d\zeta = d\beta - di\omega = d\beta - i(d\omega) = d\beta - i(i^{-1}(d\beta)) = 0,$$

thus there exists an extension of $\mathfrak{g}$ by $\mathfrak{m}$ using the cocycle $\zeta \in Z^2(\mathfrak{g}, \mathfrak{m})$.

(b) We now use this extension to construct a morphism of crossed modules. As a vector space, $\mathfrak{e} = \mathfrak{m} \oplus \mathfrak{g}$. We fix such a decomposition of $\mathfrak{e}$ and call it product coordinates.

Lemma 7.2. *The map m written in product coordinates as $m : \mathfrak{m} \oplus \mathfrak{g} \to \mathrm{im}\,(\mu) \oplus \mathfrak{g}$, $(a,x) \mapsto m(a,x) = (\mu(a), x)$ induces a morphism of crossed modules*

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathfrak{m} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{m}}"] & \mathfrak{e} \arrow[r] \arrow[d, "m"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r, "i"] & \mathfrak{m} \arrow[r, "\mu"] & \mathfrak{n} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Proof. The bracket in $\mathfrak{n}$ reads in product coordinates

$$[(a,x),(b,y)] = ([a,b] + x \cdot b - y \cdot a + \alpha(x,y), [x,y]),$$

and the one in $\mathfrak{e}$ reads

$$[(a,x),(b,y)] = ([a,b] + x \cdot b - y \cdot a + \zeta(x,y), [x,y]).$$

Now, applying μ to the first component of the lower bracket gives the first component of the upper bracket thanks to the axioms of a crossed module and to $\mu\beta = \alpha$. The map is also compatible with the actions. $\square$

(c) The third step is to prove the following lemma:

Lemma 7.3. *If a crossed module*

$$0 \to V \to \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \to \mathfrak{g} \to 0$$

admits a morphism

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathfrak{m} \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_{\mathfrak{m}}"] & \mathfrak{e} \arrow[r, "\psi"] \arrow[d, "m"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r, "i"] & \mathfrak{m} \arrow[r, "\mu"] & \mathfrak{n} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0,
\end{tikzcd}$$

then it represents the zero equivalence class.

Proof. Indeed, in case there is a morphism as indicated, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\mathrm{id}_V"] & V \arrow[r, "0"] & \mathfrak{g} \arrow[r, "\mathrm{id}_{\mathfrak{g}}"] & \mathfrak{g} \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, "\mathrm{incl}_1"] \arrow[u, "\mathrm{id}_V"] \arrow[d, "\mathrm{id}_V"] & V \oplus \mathfrak{m} \arrow[r, "0+\varphi"] \arrow[u, "\mathrm{proj}_1"] \arrow[d, "i+\mathrm{proj}_2"] & \mathfrak{e} \arrow[r, "\psi"] \arrow[u, "\psi"] \arrow[d, "m"] & \mathfrak{g} \arrow[r] \arrow[u, "\mathrm{id}_{\mathfrak{g}}"] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & V \arrow[r, "i"] & \mathfrak{m} \arrow[r, "\mu"] & \mathfrak{n} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Thus $\mu : \mathfrak{m} \to \mathfrak{n}$ represents the zero class (as equivalence class of crossed modules). We denoted by incl_1 and proj_2 once again the standard inclusion and projection maps. This concludes the proof of the Lemma, the injectivity of the map b, and thus the proof of Theorem 6.10. $\square$

Corollary 7.4. *Every crossed module is equivalent to one coming from the principal construction.*

8 Crossed modules as strict Lie 2-algebras

Here we highlight the interpretation of crossed modules of Lie algebras as strict Lie 2-algebras. A Lie 2-algebra is a categorified version of a Lie algebra. Categorification means the procedure of replacing in an algebraic structure the underlying sets and maps by categories and functors. This gives new, more sophisticated algebraic structures which are useful in physics in the context of higher gauge theory and in categorified representation theory which strives to define more powerful knot invariants, for example.

8.1 Categories and functors

A *category* $\mathcal{C}$ consists of a class (in general not a set !) of objects, $\mathrm{Ob}_{\mathcal{C}}$, together with for any pair of objects $X, Y \in \mathrm{Ob}_{\mathcal{C}}$, a set of morphisms $\mathrm{Mor}(X, Y)$ from X to Y and a composition map

$$\mathrm{comp} : \mathrm{Mor}_{\mathcal{C}}(X, Y) \times \mathrm{Mor}_{\mathcal{C}}(Y, Z) \to \mathrm{Mor}_{\mathcal{C}}(X, Z)$$

such that

(1) For any $X \in \mathrm{Ob}_{\mathcal{C}}$, there is a distinguished $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$.

(2) The composition comp is associative.

(3) The morphism id_X is a left and right unit with respect to comp.

Categories have mainly two uses: On the one hand, they provide a more abstract level to carry out some constructions and permit to better understand them. This is the category theory which deals with *large* categories like for example the category of sets `Sets`, the category of groups `Grp` or the category of k-vector spaces `Vect`. In all these three cases, the objects of the category are indicated (i.e. are sets, groups resp. k-vector spaces), while the morphisms are structure preserving maps (i.e. maps, group homomorphisms resp. k-linear maps). There are logical problems arising when dealing with large categories like the category of sets and we do not want to discuss how they are going to be solved, but they can be solved.

On the other hand, there are *small* categories which are categories where the class of objects forms a set. These are the categories which are studied as an algebraic structure: A category can be understood in this sense as a monoid with many objects (i.e. a monoid where not all elements are composable). We will take this latter point of view in the following.

As always in algebra, after introducing the objects which are studied, one has to introduce the structure preserving morphisms. For categories, these are the functors. A *functor* $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ is a prescription assigning to each object $X \in \mathrm{Ob}_{\mathcal{C}}$ an object $\mathfrak{F}(X) \in \mathrm{Ob}_{\mathcal{D}}$ and to each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ a morphism $\mathfrak{F}(f) \in \mathrm{Mor}_{\mathcal{D}}(\mathfrak{F}(X), \mathfrak{F}(Y))$ such that identities are sent to identities and compositions are sent to compositions.

Many constructions in mathematics are functorial, i.e. con be interpreted as functors between categories. For example, the free group on a set defines a functor from `Sets` to `Grp` which is related to the functor $U : \mathtt{Grp} \to \mathtt{Sets}$ which sends a group G to its underlying set $U(G)$ which is nothing but the *set* G.

8.2 Strict 2-vector spaces and 2-term complexes

Fix a field k of characteristic 0. In these notes, a 2-*vector space* V over k is simply a category object in `Vect`, the category of vector spaces. This means that V consists of a vector space of *arrows* V_1, a vector space of *objects* V_0, linear maps $V_1 \overset{s}{\underset{t}{\rightrightarrows}} V_0$ called *source* and *target*, a linear map $i : V_0 \to V_1$, called *object inclusion*, and a linear map

$$m \,:\, V_1 \times_{V_0} V_1 \to V_1,$$

which is called the *categorical composition*. These data are supposed to satisfy the usual axioms of a category.

An equivalent point of view is to view a 2-vector space as a 2-term chain complex of vector spaces $d : C_1 \to C_0$. The equivalence is spelt out as follows: One passes from a category object in $\mathtt{Vect}$ (given by $V_1 \overset{s}{\underset{t}{\rightrightarrows}} V_0$, $i : V_0 \to V_1$ etc) to a 2-term complex $d : C_1 \to C_0$ by taking $C_1 := \ker(s)$, $d := t|_{\ker(s)}$ and $C_0 = V_0$. In the reverse direction, to a given 2-term complex $d : C_1 \to C_0$, one associates $V_1 = C_0 \oplus C_1$, $V_0 = C_0$, $s(c_0, c_1) = c_0$, $t(c_0, c_1) = c_0 + d(c_1)$, and $i(c_0) = (c_0, 0)$. The only subtle point is here that the categorical composition m is already determined by $V_1 \overset{s}{\underset{t}{\rightrightarrows}} V_0$ and $i : V_0 \to V_1$. Namely, writing an arrow $c_1 =: f$ with $s(f) = x$, $t(f) = y$, i.e. $f : x \mapsto y$, one denotes the *arrow part* of f by $\vec{f} := f - i(s(f))$, and for two composable arrows $f, g \in V_1$, the composition m is then defined by

$$f \circ g \,:=\, m(f, g) \,:=\, i(x) + \vec{f} + \vec{g}.$$

Observe that we use here Baez-Crans convention on the composition, i.e. we compose from left to right (the source of $f \circ g$ is the source of f and not the source of g like in usual composition).

8.3 Strict Lie 2-algebras and crossed modules

Definition 8.1. A strict Lie 2-algebra is a category object in the category $\mathtt{Lie}$ of Lie algebras over k.

This means that it is the data of two Lie algebras, $\mathfrak{g}_0$, the *Lie algebra of objects*, and $\mathfrak{g}_1$, the *Lie algebra of arrows*, together with morphisms of Lie algebras $s, t : \mathfrak{g}_1 \to \mathfrak{g}_0$, *source* and *target*, a morphism $i : \mathfrak{g}_0 \to \mathfrak{g}_1$, the *object inclusion*, and a morphism $m : \mathfrak{g}_1 \times_{\mathfrak{g}_0} \mathfrak{g}_1 \to \mathfrak{g}_1$, the *composition of arrows*, such that the usual axioms of a category are satisfied.

Theorem 8.2. *Strict Lie 2-algebras are in one-to-one correspondence with crossed modules of Lie algebras.*

Proof. Given a Lie 2-algebra $\mathfrak{g}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{g}_0$, $i : \mathfrak{g}_0 \to \mathfrak{g}_1$, the corresponding crossed module is defined by

$$\mu := t|_{\ker(s)} \,:\, \mathfrak{m} := \ker(s) \to \mathfrak{n} := \mathfrak{g}_0.$$

The action of $\mathfrak{n}$ on $\mathfrak{m}$ is given by

$$n \cdot m \,:=\, [i(n), m],$$

for $n \in \mathfrak{n}$ and $m \in \mathfrak{m}$ (where the bracket is taken in $\mathfrak{g}_1$). This is well defined and an action by derivations. Axiom (a) follows from

$$\mu(n \cdot m) = \mu([i(n), m]) = [\mu \circ i(n), \mu(m)] = [n, \mu(m)].$$

Axiom (b) follows from

$$\mu(m) \cdot m' = [i \circ \mu(m), m'] = [i \circ t(m), m'] = [m + r, m'] = [m, m']$$

by writing $i \circ t(m) = m + r$ with $r \in \ker(t)$ and by using that $\ker(t)$ and $\ker(s)$ in a Lie 2-algebra commute. This is shown in Lemma 8.3 after the proof.

On the other hand, given a crossed module of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$, associate to it

$$\mathfrak{m} \rtimes \mathfrak{n} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{n}\,, \quad i : \mathfrak{n} \to \mathfrak{m} \rtimes \mathfrak{n}$$

with $s(m, n) = n$, $t(m, n) = \mu(m) + n$, $i(n) = (0, n)$, where the semi-direct product Lie algebra $\mathfrak{m} \rtimes \mathfrak{n}$ is built from the given action of $\mathfrak{n}$ on $\mathfrak{m}$. Let us emphasize that $\mathfrak{m} \rtimes \mathfrak{n}$ is built from the Lie algebra $\mathfrak{n}$ and the $\mathfrak{n}$-module $\mathfrak{m}$; the bracket of $\mathfrak{m}$ does not intervene here. The composition of arrows is already encoded in the underlying structure of 2-vector space, as remarked in the previous subsection. In the second lemma below, we show that the composition is a morphism of Lie algebras. $\square$

Lemma 8.3. $[\ker(s), \ker(t)] = 0$ *in a Lie* 2*-algebra.*

Proof. The fact that the composition of arrows is a homomorphism of Lie algebras gives the following "middle four exchange" (or functoriality) property

$$[g_1, g_2] \circ [f_1, f_2] \;=\; [g_1 \circ f_1, g_2 \circ f_2]$$

for composable arrows $f_1, f_2, g_1, g_2 \in \mathfrak{g}_1$. Now suppose that $g_1 \in \ker(s)$ and $f_2 \in \ker(t)$. Then denote by f_1 and by g_2 the identity (w.r.t. the composition) in $0 \in \mathfrak{g}_0$. As these are identities, we have $g_1 = g_1 \circ f_1$ and $f_2 = g_2 \circ f_2$. On the other hand, i is a morphism of Lie algebras and sends $0 \in \mathfrak{g}_0$ to the $0 \in \mathfrak{g}_1$. Therefore we may conclude

$$[g_1, f_2] \;=\; [g_1 \circ f_1, g_2 \circ f_2] \;=\; [g_1, g_2] \circ [f_1, f_2] \;=\; 0.$$

$\square$

Lemma 8.4. *Given a crossed module of Lie algebras* $\mu : \mathfrak{m} \to \mathfrak{n}$*, the composition of the underlying* 2*-vector space of the precategory* $\mathfrak{m} \rtimes \mathfrak{n} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{n}$ *is a morphism of Lie algebras.*

Proof. Every morphism $f = (m, n) \in \mathfrak{m} \rtimes \mathfrak{n}$ is described in the underlying 2-vector space by its starting point $s(f) = s(m, n) = n$ and its arrow part

$\vec{f} = f - i(s(f)) = (m, 0)$. We have to show the double-four-echange law, i.e. for composable arrows f_1, f_2, g_1, g_2 we need to show

$$[g_1 \circ f_1, g_2 \circ f_2] = [g_1, g_2] \circ [f_1, f_2].$$

Here it is understood that $g_i : x_i \mapsto y_i$ and $f_i : y_i \mapsto z_i$ for $i = 1, 2$. The fact that the two are composable means that $t(g_i) = y_i = s(f_i)$ for $i = 1, 2$. Now translate all this into elements of the semidirect product. We call $g_i = (m_i, n_i)$ and $f_i = (m'_i, n'_i)$ for $i = 1, 2$, and the composability means now $t(m_i, n_i) = \mu(m_i) + n_i = n'_i = s(f_i)$ for $i = 1, 2$.

We compute

$$\begin{aligned}
[g_1 \circ f_1, g_2 \circ f_2] &= [i(s(g_1)) + \vec{g_1} + \vec{f_1}, i(s(g_2)) + \vec{g_2} + \vec{f_2}] \\
&= [(0, n_1) + (m_1, 0) + (m'_1, 0), (0, n_2) + (m_2, 0) + (m'_2, 0)] \\
&= [(m_1 + m'_1, n_1), (m_2 + m'_2, n_2)] \\
&= ([m_1 + m'_1, m_2 + m'_2] + n_1 \cdot (m_2 + m'_2) - n_2 \cdot (m_1 + m'_1), [n_1, n_2]).
\end{aligned}$$

This is to compare with

$$\begin{aligned}
[g_1, g_2] \circ [f_1, f_2] &= i(s([g_1, g_2])) + \overrightarrow{[g_1, g_2]} + \overrightarrow{[f_1, f_2]} \\
&= ([m_1, m_2] + n_1 \cdot m_2 - n_2 \cdot m_1 + \\
&+ [m'_1, m'_2] + n'_1 \cdot m'_2 - n'_2 \cdot m'_1, [n_1, n_2])
\end{aligned}$$

because

$$\begin{aligned}
[g_1, g_2] &= [(m_1, n_1), (m_2, n_2)] \\
&= ([m_1, m_2] + n_1 \cdot m_2 - n_2 \cdot m_1, [n_1, n_2])
\end{aligned}$$

and

$$\overrightarrow{[g_1, g_2]} = ([m_1, m_2] + n_1 \cdot m_2 - n_2 \cdot m_1, 0)$$

and

$$\overrightarrow{[f_1, f_2]} = ([m'_1, m'_2] + n'_1 \cdot m'_2 - n'_2 \cdot m'_1, 0).$$

The equality of these two expressions now follows from the following computation, where we use the $\mu(m_i) + n_i = n'_i$ for $i = 1, 2$ and property (b) (i.e. $\mu(m) \cdot m' = [m, m']$):

$$\begin{aligned}
[m'_1, m_2] + [m_1, m'_2] + n_1 \cdot m'_2 - n_2 \cdot m'_1 - n'_1 \cdot m_2 + n'_2 \cdot m'_1 &= \\
= [m'_1, m_2] + [m_1, m'_2] + n_1 \cdot m'_2 - n_2 \cdot m'_1 - (n_1 + \mu(m_1)) \cdot m'_2 + (n_2 + \mu(m_2)) \cdot m'_1 &= \\
= [m'_1, m_2] + [m_1, m'_2] - \mu(m_1) \cdot m_2 + \mu(m_2) \cdot m'_1 &= 0.
\end{aligned}$$

□

Remark 8.5. (a) It is implicit in the previous proof that starting from a crossed module $\mu : \mathfrak{m} \to \mathfrak{n}$, passing to the Lie 2-algebra $\mathfrak{g}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{g}_0$,

$i : \mathfrak{g}_0 \to \mathfrak{g}_1$ (and thus forgetting the bracket on $\mathfrak{m}$!), one may finally reconstruct the bracket on $\mathfrak{m}$. This is due to the fact that the bracket on $\mathfrak{m}$ is encoded in the action by the property (b) of a crossed module

$$[m, m'] = \mu(m) \cdot m'.$$

(b) It is shown in Corollary 7.4 that for a given third cohomology class (or, equivalently, for a given equivalence class of crossed modules), there is a crossed module $\mu : \mathfrak{m} \to \mathfrak{n}$ representing this class such that the bracket on $\mathfrak{m}$ is trivial.

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