Project 1: A different proof of Gerstenhaber’s theorem

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The goal of this project is to show Gerstenhaber’s theorem using relative crossed modules. This approach is due to Kassel and Loday in Fix a surjective morphism of Lie algebras \( \pi : n \to g \) and a \( g \)-module \( V \).

**Definition 0.1.** Two crossed modules \( \mu_i : m_i \to n \) for \( i = 1, 2 \) are called relatively equivalent if there exists a homomorphism of Lie algebras \( \varphi : n_1 \to n_2 \) such that the diagram

\[
\begin{align*}
0 & \to V \overset{i_1}{\to} m_1 \overset{\mu_1}{\to} n \overset{\pi}{\to} g \overset{id_g}{\to} 0 \\
0 & \to V \overset{i_2}{\to} m_2 \overset{\mu_2}{\to} n \overset{\pi}{\to} g \overset{id_g}{\to} 0
\end{align*}
\]

is commutative and such that \( \varphi \) is equivariant for the actions \( \eta_i \) of \( n \) on \( m_i \) for \( i = 1, 2 \).

**Remark 0.2.** By the Five Lemma, the homomorphism \( \varphi \) is necessarily an isomorphism. Therefore, it is clear that relative equivalence is an equivalence relation.

The main theorem on relative crossed modules reads then:

**Theorem 0.3.** There is a natural bijection

\[
\text{crmod}(g, n, V) \cong H^3(g, n, V)
\]

between the set of relative equivalence classes of crossed modules \( \mu : m \to n \) with fixed quotient morphism \( \pi : n \to g \) and fixed \( g \)-module \( V = \ker(\mu) \), and the third relative cohomology group \( H^3(g, n, V) \).

**Proof.** Let \( \mu : m \to n \) be a crossed module with quotient morphism \( \pi : n \to g \) and with \( \ker(\mu) \) identified with \( V \) as a \( g \)-module. Kassel and Loday associate to the crossed module \( \mu \) a 2-cocycle \( f \) with values in \( V \) associated to a section \( s : g \to n \) of \( \pi \) and a section \( \sigma : \text{im}(\mu) = \ker(\pi) \to m \) of \( \mu \). For all \( x, y \in g \), they set

\[
g(x, y) = \sigma([s(x), s(y)] - s[x, y]) \in m.
\]
This corresponds to our $\beta(x,y)$. Furthermore, they set for all $n \in \mathfrak{n}$

$$
\Psi(n) = \sigma(n - s \circ \pi(n)) \in \mathfrak{m}.
$$

With these notations, they define

$$
f(n,n') = g(\pi(n), \pi(n')) - n' \cdot \Psi(n) + n \cdot \Psi(n') - [\Psi(n), \Psi(n')] - \Psi[n,n'].
$$

This 2-cochain is defined on $\mathfrak{n}$. Show that $f$ has values in $\mathfrak{V}$.

Kassel and Loday need a relative cocycle. The complex of relative Lie algebra cohomology is by definition the following quotient complex

$$
0 \to C^*(\mathfrak{g}, \mathfrak{V}) \xrightarrow{\pi_*} C^*(\mathfrak{n}, \mathfrak{V}) \xrightarrow{\kappa_*} C^*(\mathfrak{g}, \mathfrak{n}, \mathfrak{V}) \to 0.
$$

The relative cocycle they associate to the crossed module $\mu$ is defined to be $\kappa^* f \in C^2(\mathfrak{g}, \mathfrak{n}, \mathfrak{V})$. The cohomology in $C^2(\mathfrak{g}, \mathfrak{n}, \mathfrak{V})$ is denoted $H^3(\mathfrak{g}, \mathfrak{n}, \mathfrak{V})$. In order to show that $\kappa^* f$ is a cocycle, Kassel and Loday introduce a cochain $k$ defined by

$$
k(x,y,z) = \sum_{\text{cycl}} g(x, [y,z]) + \sum_{\text{cycl}} s(x) \cdot g(y,z) \in \mathfrak{V}.
$$

Now the situation is the following:

$$
\begin{array}{ccc}
C^2(\mathfrak{g}, \mathfrak{V}) & \xrightarrow{\pi_*} & C^2(\mathfrak{n}, \mathfrak{V}) & \xrightarrow{\kappa_*} & C^2(\mathfrak{g}, \mathfrak{n}, \mathfrak{V}) \\
\downarrow d & & \downarrow d & & \downarrow d \\
C^3(\mathfrak{g}, \mathfrak{V}) & \xrightarrow{\pi_*} & C^3(\mathfrak{n}, \mathfrak{V}) & \xrightarrow{\kappa_*} & C^3(\mathfrak{g}, \mathfrak{n}, \mathfrak{V})
\end{array}
$$

Show that $df = \pi^* k$. This identity then implies that $d\kappa^* f = \kappa^* df = \kappa^* \pi^* f = 0$, and therefore $\kappa^* f$ is a cocycle.

What we have done so far can be resumed in the existence of a well-defined map

$$
crmod(\mathfrak{g}, \mathfrak{n}, \mathfrak{V}) \to H^3(\mathfrak{g}, \mathfrak{n}, \mathfrak{V}), \ [\mu : \mathfrak{m} \to \mathfrak{n}] \mapsto [\kappa^* f].
$$

Conversely, suppose given a cocycle in $C^2(\mathfrak{g}, \mathfrak{n}, \mathfrak{V})$ which we lift to a cochain $f \in C^2(\mathfrak{n}, \mathfrak{V})$. As $\kappa^* f$ is a cocycle, we have a cochain $k \in C^3(\mathfrak{g}, \mathfrak{V})$ such that $df = \pi^* k$. In particular, the restriction of $f$ to $\ker(\pi) =: \mathfrak{l}$ gives a a cocycle in $C^2(\mathfrak{l}, \mathfrak{V})$. We get thus a Lie algebra structure on the direct sum $\mathfrak{m} = \mathfrak{V} \oplus \mathfrak{l}$ which makes it a central extension using the bracket

$$
[(z,l), (z',l')] = (f(l_1,l_2), [l_1, l_2]).
$$

Restriction onto $\mathfrak{n} \times \mathfrak{l}$, we obtain from $f$ an action of $\mathfrak{n}$ on $\mathfrak{m}$ by the formula

$$
n \cdot (z,l) = (\pi(n) \cdot z + f(n,l), [n,l]).
$$

Check that with these data, the map $\mu : \mathfrak{m} \to \mathfrak{n}$, given by $(z,l) \mapsto l$, is a crossed module. Show that the addition of a coboundary to $f$ does not affect the (relative) equivalence class of this crossed module.
We thus get a well-defined map

\[ H^3(\mathfrak{g}, n, V) \to \text{crmod}(\mathfrak{g}, n, V), \quad [\kappa^* f] \mapsto [\mu : m \to n]. \]

By construction, we obtain as associated cohomology class to this crossed module the class of \( f \). In the other direction, show that the two maps also compose to the identity. \(\square\)

**Remark 0.4.** The relation of the relative class \([\kappa^* f] \in H^3(\mathfrak{g}, n, V)\) to the absolute class \([d\tilde{\theta}] \in H^3(\mathfrak{g}, V)\) is given by the connecting homomorphism in the long exact sequence in cohomology associated to the short exact sequence of complexes

\[ 0 \to C^*(\mathfrak{g}, V) \xrightarrow{\pi^*} C^*(n, V) \xrightarrow{\kappa^*} C^*(\mathfrak{g}, n, V) \to 0. \]

Indeed, by definition of the connecting homomorphism \(\partial\), the image \(\partial(\kappa^* f)\) is obtained by first lifting \(\kappa^* f\) to a cochain in \(C^3(n, V)\), for which we may take \( f \), then by taking its coboundary \( df \) and finally by identifying \( df \) with the image \(\pi^* k\) of some element \( k \in C^3(\mathfrak{g}, V)\). By definition, \([\partial(\kappa^* f)] = [k]\). We see that the connecting homomorphism \(\partial\) sends Kassel-Loday’s relative class to the absolute class.

In order to state this once again more neatly, introduce the forgetting map \(D : \text{crmod}(\mathfrak{g}, n, V) \to \text{crmod}(\mathfrak{g}, V)\) which forgets the fixed quotient morphism \(\pi : n \to \mathfrak{g}\). It is well defined. Then we have a commutative diagram

\[
\begin{array}{ccc}
H^3(\mathfrak{g}, n, V) & \cong & \text{crmod}(\mathfrak{g}, n, V) \\
\downarrow{\partial} & & \downarrow{\partial} \\
H^3(\mathfrak{g}, V) & \cong & \text{crmod}(\mathfrak{g}, V)
\end{array}
\]

**Remark 0.5.** In fact, Theorem 0.3 even implies Gerstenhaber’s Theorem.

Indeed, given an epimorphism \(\pi : n \to \mathfrak{g}\) and a \(\mathfrak{g}\)-module \(V\), consider the long exact sequence in cohomology induced by the short exact sequence of complexes

\[ 0 \to C^*(\mathfrak{g}, V) \to C^*(n, V) \to C^*(\mathfrak{g}, n, V) \to 0. \]

There is furthermore an exact sequence

\[ \text{Ext}(n, V) \to \text{crmod}(\mathfrak{g}, n, V) \to \text{crmod}(\mathfrak{g}, V) \to \text{crmod}(n, V), \]

where \(V\) is viewed as an \(n\)-module via \(\pi : n \to \mathfrak{g}\). Together, we have an exact ladder

\[
\begin{array}{cccccc}
\ldots & H^2(\mathfrak{g}, V) & \longrightarrow & H^3(\mathfrak{g}, n, V) & \longrightarrow & H^3(\mathfrak{g}, V) & \longrightarrow & \ldots \\
\downarrow & & & \downarrow & & \downarrow & & \\
\ldots & \text{Ext}(n, V) & \longrightarrow & \text{crmod}(\mathfrak{g}, n, V) & \longrightarrow & \text{crmod}(\mathfrak{g}, V) & \longrightarrow & \ldots
\end{array}
\]

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Such an exact ladder exists for each choice of $V$ and $\pi : n \to g$. Now suppose that $V$ is an injective $g$-module. Then $H^3(g, V) = 0$. Show that the isomorphism between relative crossed modules and relative 3-cohomology implies that in this case $\text{crmod}(g, V) = 0$.

For the general case, embed $V$ into an injective $g$-module $I$ with quotient $Q$:

$$0 \to V \to I \to Q \to 0.$$  

This short exact sequence of coefficients induces long exact sequences both in cohomology and gives an exact ladder

$$\cdots \to H^2(g, Q) \to H^3(g, V) \to H^3(g, I) \to \cdots$$

$$\cdots \to \text{Ext}(g, Q) \to \text{crmod}(g, V) \to \text{crmod}(g, I) \to \cdots$$

Here we have $H^3(g, I) = 0$ and $\text{crmod}(g, I) = 0$ by the preceding, and the map $\text{Ext}(g, Q) \to H^2(g, Q)$ is an isomorphism. Conclude.