# Project 1: A different proof of Gerstenhaber's theorem

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The goal of this project is to show Gerstenhaber's theorem using *relative crossed modules*. This approach is due to Kassel and Loday in Fix a surjective morphism of Lie algebras $\pi : \mathfrak{n} \to \mathfrak{g}$ and a $\mathfrak{g}$-module $V$.

**Definition 0.1.** Two crossed modules $\mu_i : \mathfrak{m}_i \to \mathfrak{n}$ for $i = 1, 2$ are called *relatively equivalent* if there exists a homomorphism of Lie algebras $\varphi : \mathfrak{n}_1 \to \mathfrak{n}_2$ such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V & \xrightarrow{i_1} & \mathfrak{m}_1 & \xrightarrow{\mu_1} & \mathfrak{n} & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{n}}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{g}}} & & \\ 0 & \longrightarrow & V & \xrightarrow{i_2} & \mathfrak{m}_2 & \xrightarrow{\mu_2} & \mathfrak{n} & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

is commutative and such that $\varphi$ is equivariant for the actions $\eta_i$ of $\mathfrak{n}$ on $\mathfrak{m}_i$ for $i = 1, 2$.

**Remark 0.2.** By the Five Lemma, the homomorphism $\varphi$ is necessarily an isomorphism. Therefore, it is clear that relative equivalence is an equivalence relation.

The main theorem on relative crossed modules reads then:

**Theorem 0.3.** *There is a natural bijection*

$$\mathrm{crmod}(\mathfrak{g}, \mathfrak{n}, V) \cong H^3(\mathfrak{g}, \mathfrak{n}, V)$$

*between the set of relative equivalence classes of crossed modules $\mu : \mathfrak{m} \to \mathfrak{n}$ with fixed quotient morphism $\pi : \mathfrak{n} \to \mathfrak{g}$ and fixed $\mathfrak{g}$-module $V = \ker(\mu)$, and the third relative cohomology group $H^3(\mathfrak{g}, \mathfrak{n}, V)$.*

*Proof.* Let $\mu : \mathfrak{m} \to \mathfrak{n}$ be a crossed module with quotient morphism $\pi : \mathfrak{n} \to \mathfrak{g}$ and with $\ker(\mu)$ identified with $V$ as a $\mathfrak{g}$-module. Kassel and Loday associate to the crossed module $\mu$ a 2-cochain $f$ with values in $V$ associated to a section $s : \mathfrak{g} \to \mathfrak{n}$ of $\pi$ and a section $\sigma : \mathrm{im}\,(\mu) = \ker(\pi) \to \mathfrak{m}$ of $\mu$. For all $x, y \in \mathfrak{g}$, they set

$$g(x, y) = \sigma([s(x), s(y)] - s[x, y]) \in \mathfrak{m}.$$

This corresponds to our $\beta(x,y)$. Furthermore, they set for all $n \in \mathfrak{n}$

$$\Psi(n) = \sigma(n - s \circ \pi(n)) \in \mathfrak{m}.$$

With these notations, they define

$$f(n,n') = g(\pi(n),\pi(n')) - n' \cdot \Psi(n) + n \cdot \Psi(n') - [\Psi(n),\Psi(n')] - \Psi[n,n'].$$

This 2-cochain in defined on $\mathfrak{n}$. Show that $f$ has values in $V$.

Kassel and Loday need a relative cocycle. The complex of relative Lie algebra cohomology is by definition the following quotient complex

$$0 \to C^*(\mathfrak{g},V) \stackrel{\pi^*}{\to} C^*(\mathfrak{n},V) \stackrel{\kappa^*}{\to} C^*(\mathfrak{g},\mathfrak{n},V) \to 0.$$

The relative cocycle they associate to the crossed module $\mu$ is defined to be $\kappa^* f \in C^2(\mathfrak{g},\mathfrak{n},V)$. The cohomology in $C^2(\mathfrak{g},\mathfrak{n},V)$ is denoted $H^3(\mathfrak{g},\mathfrak{n},V)$. In order to show that $\kappa^* f$ is a cocycle, Kassel and Loday introduce a cochain $k$ defined by

$$k(x,y,z) = \sum_{\text{cycl.}} g(x,[y,z]) + \sum_{\text{cycl.}} s(x) \cdot g(y,z) \in V.$$

Now the situation is the following:

$$\begin{array}{ccccc} C^2(\mathfrak{g},V) & \xrightarrow{\pi^*} & C^2(\mathfrak{n},V) & \xrightarrow{\kappa^*} & C^2(\mathfrak{g},\mathfrak{n},V) \\ \downarrow d & & \downarrow d & & \downarrow d \\ C^3(\mathfrak{g},V) & \xrightarrow{\pi^*} & C^3(\mathfrak{n},V) & \xrightarrow{\kappa^*} & C^3(\mathfrak{g},\mathfrak{n},V) \end{array}$$

Show that $df = \pi^* k$. This identity then implies that $d\kappa^* f = \kappa^* df = \kappa^* \pi^* f = 0$, and therefore $\kappa^* f$ is a cocycle.

What we have done so far can be resumed in the existence of a well-defined map

$$\text{crmod}(\mathfrak{g},\mathfrak{n},V) \to H^3(\mathfrak{g},\mathfrak{n},V),\ \ [\mu : \mathfrak{m} \to \mathfrak{n}] \mapsto [\kappa^* f].$$

Conversely, suppose given a cocycle in $C^2(\mathfrak{g},\mathfrak{n},V)$ which we lift to a cochain $f \in C^2(\mathfrak{n},V)$. As $\kappa^* f$ is a cocycle, we have a cochain $k \in C^3(\mathfrak{g},V)$ such that $df = \pi^* k$. In particular, the restriction of $f$ to $\ker(\pi) =: \mathfrak{l}$ gives a a cocycle in $C^2(\mathfrak{l},V)$. We get thus a Lie algebra structure on the direct sum $\mathfrak{m} = V \oplus \mathfrak{l}$ which makes it a central extension using the bracket

$$[(z,l),(z',l')] = (f(l_1,l_2),[l_1,l_2]).$$

Restriction onto $\mathfrak{n} \times \mathfrak{l}$, we obtain from $f$ an action of $\mathfrak{n}$ on $\mathfrak{m}$ by the formula

$$n \cdot (z,l) = (\pi(n) \cdot z + f(n,l),[n,l]).$$

Check that with these data, the map $\mu : \mathfrak{m} \to \mathfrak{n}$, given by $(z,l) \mapsto l$, is a crossed module. Show that the addition of a coboundary to $f$ does not affect the (relative) equivalence class of this crossed module.

We thus get a well-defined map

$$H^3(\mathfrak{g},\mathfrak{n},V)\to \mathrm{crmod}(\mathfrak{g},\mathfrak{n},V),\ \ [\kappa^* f]\mapsto [\mu:\mathfrak{m}\to\mathfrak{n}].$$

By construction, we obtain as associated cohomology class to this crossed module the class of $f$. In the other direction, show that the two maps also compose to the identity. □

**Remark 0.4.** The relation of the relative class $[\kappa^* f]\in H^3(\mathfrak{g},\mathfrak{n},V)$ to the absolute class $[d_S\tilde{\theta}]\in H^3(\mathfrak{g},V)$ is given by the connecting homomorphism in the long exact sequence in cohomology associated to the short exact sequence of complexes

$$0\to C^*(\mathfrak{g},V)\stackrel{\pi^*}{\to} C^*(\mathfrak{n},V)\stackrel{\kappa^*}{\to} C^*(\mathfrak{g},\mathfrak{n},V)\to 0.$$

Indeed, by definition of the connecting homomorphism $\partial$, the image $\partial(\kappa^* f)$ is obtained by first lifting $\kappa^* f$ to a cochain in $C^2(\mathfrak{n},V)$, for which we may take $f$, then by taking its coboundary $df$ and finally by identifying $df$ with the image $\pi^* k$ of some element $k\in C^3(\mathfrak{g},V)$. By definition, $[\partial(\kappa^* f)]$ is then set to be $[\partial(\kappa^* f)]=[k]$. We see that the connecting homomorphism $\partial$ sends Kassel-Loday's relative class to the absolute class.

In order to state this once again more neatly, introduce the forgetting map $D:\mathrm{crmod}(\mathfrak{g},\mathfrak{n},V)\to\mathrm{crmod}(\mathfrak{g},V)$ which forgets the fixed quotient morphism $\pi:\mathfrak{n}\to\mathfrak{g}$. It is well defined. Then we have a commutative diagram

$$\begin{array}{ccc} H^3(\mathfrak{g},\mathfrak{n},V) & \xrightarrow{\cong} & \mathrm{crmod}(\mathfrak{g},\mathfrak{n},V) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle D} \\ H^3(\mathfrak{g},V) & \xrightarrow{\cong} & \mathrm{crmod}(\mathfrak{g},V) \end{array}$$

**Remark 0.5.** In fact, Theorem 0.3 even implies Gerstenhaber's Theorem.

Indeed, given an epimorphism $\pi:\mathfrak{n}\to\mathfrak{g}$ and a $\mathfrak{g}$-module $V$, consider the long exact sequence in cohomology induced by the short exact sequence of complexes

$$0\to C^*(\mathfrak{g},V)\to C^*(\mathfrak{n},V)\to C^*(\mathfrak{g},\mathfrak{n},V)\to 0.$$

There is furthermore an exact sequence

$$\mathrm{Ext}\,(\mathfrak{n},V)\to \mathrm{crmod}(\mathfrak{g},\mathfrak{n},V)\to\mathrm{crmod}(\mathfrak{g},V)\to\mathrm{crmod}(\mathfrak{n},V),$$

where $V$ is viewed as an $\mathfrak{n}$-module via $\pi:\mathfrak{n}\to\mathfrak{g}$. Together, we have an exact ladder

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & H^2(\mathfrak{n},V) & \longrightarrow & H^3(\mathfrak{g},\mathfrak{n},V) & \longrightarrow & H^3(\mathfrak{g},V) & \longrightarrow & \ldots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \ldots & \longrightarrow & \mathrm{Ext}\,(\mathfrak{n},V) & \longrightarrow & \mathrm{crmod}(\mathfrak{g},\mathfrak{n},V) & \longrightarrow & \mathrm{crmod}(\mathfrak{g},V) & \longrightarrow & \ldots \end{array}$$

Such an exact ladder exists for each choice of $V$ and $\pi : \mathfrak{n} \to \mathfrak{g}$. Now suppose that $V$ is an injective $\mathfrak{g}$-module. Then $H^3(\mathfrak{g}, V) = 0$. Show that the isomorphism between relative crossed modules and relative 3-cohomology implies that in this case $\mathrm{crmod}(\mathfrak{g}, V) = 0$.

For the general case, embed $V$ into an injective $\mathfrak{g}$-module $I$ with quotient $Q$:

$$0 \to V \to I \to Q \to 0.$$

This short exact sequence of coefficients induces long exact sequences both in cohomology and gives an exact ladder

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^2(\mathfrak{g}, Q) & \longrightarrow & H^3(\mathfrak{g}, V) & \longrightarrow & H^3(\mathfrak{g}, I) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathrm{Ext}\,(\mathfrak{g}, Q) & \longrightarrow & \mathrm{crmod}(\mathfrak{g}, V) & \longrightarrow & \mathrm{crmod}(\mathfrak{g}, I) & \longrightarrow & \dots
\end{array}$$

Here we have $H^3(\mathfrak{g}, I) = 0$ and $\mathrm{crmod}(\mathfrak{g}, I) = 0$ by the preceding, and the map $\mathrm{Ext}\,(\mathfrak{g}, Q) \to H^2(\mathfrak{g}, Q)$ is an isomorphism. Conclude.