Project 2: A crossed module representing the Godbillon-Vey class

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We consider the Godbillon-Vey cocycle θ whose class $[\theta] \in H^3(\text{Vect}\,(S^1), \mathbb{C})$ generates the third (complex valued) cohomology of the Lie algebra of vector fields on the line W_1.

The Lie algebra W_1 of (complex valued) polynomial vector fields on the line, in one formal variable z. As a complex vector space,

$$W_1 = \bigoplus_{n \geq -1} \mathbb{C} z^{n+1} \frac{d}{dz},$$

and the bracket is given by

$$\left[z^{m+1}\frac{d}{dz}, z^{n+1}\frac{d}{dz}\right] = (n-m) z^{n+m+1}\frac{d}{dz}.$$

Sometimes, one writes these generators as $e_n := z^{n+1}\frac{d}{dz}$ and then the bracket takes the form $[e_m, e_n] = (n-m)e_{n+m}$. Usually, when one speaks about *formal* vector fields, the coefficient functions are supposed to be formal series and not polynomials, so this corresponds then to $\widehat{W_1} = \Pi_{n\geq -1}\mathbb{C}z^{n+1}\frac{d}{dz}$ and the bracket is also written formally as

$$\left[f(z)\frac{d}{dz}, g(z)\frac{d}{dz}\right] = (fg' - gf')(z)\frac{d}{dz},$$

for formal series $f, g \in \mathbb{C}[[z]]$. Usually, one identifies a vector field (polynomial or formal) with its coefficient function and writes simply f for $f\frac{d}{dz}$.

The following defines a one parameter family of 3-cocycles on W_1 (and $\widehat{W_1}$):

$$\theta_z(f,g,h) := \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}(z).$$

Usually one takes as Godbillon-Vey cocycle the evaluation in 0, i.e. $\theta := \theta_0$, but actually all θ_z are cohomologuous.

In the same way as $\widehat{W_1}$ represents formal vector fields, there are formal functions and formal differential forms. More generally, let F_λ be the *space of (polynomial) λ-densities*, i.e. as a complex vector space $F_\lambda = \bigoplus_{n\geq -1}\mathbb{C}z^{n+1}(dz)^\lambda$ and F_λ becomes a W_1-module by setting

$$f(z)\frac{d}{dz}\cdot g(z)(dz)^\lambda = (fg' + \lambda g f')(z)(dz)^\lambda.$$

In the same way, the space of formal λ-densities is denoted by $\widehat{F}_\lambda$. Observe that $\widehat{F}_0$ are formal functions on the complex line $\mathbb{C}$, $\widehat{F}_1$ are formal 1-forms, and $\widehat{F}_{-1} = \widehat{W}_1$ as a $\widehat{W}_1$-module (where $\widehat{W}_1$ is regarded as a $\widehat{W}_1$-module using the adjoint action).

It is easy to verify that the formal de Rham sequence

$$0 \to \mathbb{C} \to \widehat{F}_0 \stackrel{d_{\rm dR}}{\to} \widehat{F}_1 \to 0 \tag{1}$$

is a short exact sequence of $\widehat{W}_1$-modules.

There is another cocycle which plays a role in the construction. Namely, let $\alpha \in Z^2(\widehat{W}_1, \widehat{F}_1)$ be defined by

$$\alpha(f,g) := \left| \begin{array}{cc} f' & g' \\ f'' & g'' \end{array} \right| (z)(dz)^1.$$

Remark 0.1. This cocycle is the integrand of the *Gelfand-Fuchs cocycle*

$$\omega(f,g) := \int_{S^1} \left| \begin{array}{cc} f' & g' \\ f'' & g'' \end{array} \right| (t)dt$$

whose class $[\omega]$ generates $H^2(\mathrm{Vect}\,(S^1), \mathbb{C})$. The Gelfand-Fuks cocycle also defines a central extension of $\mathrm{Vect}\,(S^1)$ which gives again (up to a factor) the Virasoro algebra.

The key relation between the two cocycles θ_z and α is described in the following lemma. In its statement, θ_z is viewed as a function in z to which one applies the de Rham differential and obtains a 1-form. On the other hand, to the cocycle $\alpha \in Z^2(\widehat{W}_1, \widehat{F}_1)$, one may apply the Chevalley-Eilenberg differential d (corresponding to coefficients in the trivial $\widehat{W}_1$-module $\mathbb{C}$), and the result is non trivial.

Lemma 0.2. $d_{\rm dR}\theta_z = d\alpha$.

Proof. Exercise. □

Corollary 0.3. *The connecting homomorphism induced by the short exact sequence* (1) *sends* α *to (the negative of)* θ_0*, i.e.*

$$\partial\alpha = -\theta_0.$$

Proof. Exercise. □

Corollary 0.4. *The de Rham sequence (1) and the abelian extension of* $\widehat{W}_1$ *by* $\widehat{F}_1$ *using the* 2*-cocycle* α *fit together to give a crossed module of Lie algebras*

$$0 \to \mathbb{C} \to \widehat{F}_0 \to \widehat{F}_1 \times_\alpha \widehat{W}_1 \to \widehat{W}_1 \to 0.$$

This crossed module represents the Godbillon-Vey class in $H^3(\widehat{W}_1, \mathbb{C})$*.*

Proof. The statement follows immediately from the construction theorem in the lectures and the above Corollary. □

Remark 0.5. It is possible to construct similar crossed modules for the corresponding Godbillon-Vey classes in related Lie algebras like W_1, $\mathrm{Vect}\,(S^1)$ or even $\mathrm{Hol}(\Sigma_k)$, the Lie algebra of holomorphic vector fields on the open Riemann surface $\Sigma_k := \Sigma \setminus \{p_1, \ldots, p_k\}$ where Σ is a compact connected Riemann surface, $p_1, \ldots, p_k$ are pairwise distinct points of Σ and $k \geq 1$, and also $\mathrm{Vect}_{1,0}(\Sigma)$ or $\mathrm{Vect}_{0,1}(\Sigma)$, the Lie algebras of smooth vector fields of type $(1,0)$ (resp. of type $(0,1)$) on the compact connected Riemann surface Σ.

Let us here only comment on $\mathrm{Vect}\,(S^1)$. Indeed, the de Rham sequence for S^1 instead of the line is not suitable for the construction, as it reads

$$0 \to \mathbb{R} \to C^\infty(S^1) \to \Omega^1(S^1) \to H^1(S^1, \mathbb{R}) \to 0,$$

and thus has four terms instead of three. The way out is to lift vector fields on the circle to its universal covering which is the real line, and to make them in this way act on the de Rham sequence on the line. This idea leads to the suitable crossed module representing the Godbillon-Vey cocycle for $\mathrm{Vect}\,(S^1)$, and the same idea also let to the crossed module for the group of diffeomorphisms on the circle.

Remark 0.6. Let us comment on the construction of the Gelfand-Fuchs cocycle

$$\omega(f, g) = \int_{S^1} \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} (t)dt$$

as a fiber integral of the Godbillon-Vey cocycle θ_z. The Godbillon-Vey cocycle θ_z is here seen as a formal function on the line with values in Lie algebra 1-cocycles of $\widehat{W}_1$. As such we may apply the de Rham differential to θ_z, and the above lemma shows that $d_{\mathrm{dR}}\theta_z = d\alpha$.

In order to define now the *fiber integral* $\int_{S^1} \alpha(t)dt$, observe that vector fields on the circle may be Taylor expanded in some point $t \in S^1$ in order to give formal vector fields. As the circle S^1 has a trivial tangent bundle, the expansions in the different points $t \in S^1$ form a smooth function in $t \in S^1$ with values in formal vector fields. The expression $\int_{S^1} \alpha(t)dt$ takes these expansions at t, inserts them into α and integrates the obtained function over S^1. One sees that this fiber integral gives here the Gelfand-Fuchs cocycle.

The cocycle identity for the fiber integral follows directly from the formula $d_{\mathrm{dR}}\theta_z = d\alpha$. Indeed,

$$d \int_{S^1} \alpha(t)dt = \int_{S^1} d\alpha(t)dt = \int_{S^1} d_{\mathrm{dR}}\theta_z = 0$$

by Stokes' Theorem.